THEORY OF DETERMINANTS

London: C. J. CLAY AND SONS,
CAMBRIDGE UNIVERSITY PRESS WAREHOUSE,
AVE MARIA LANE.
Glasgow: 50, WELLINGTON STREET.

Leipzig: F. A. BROCKHAUS.
New York: THE MACMILLAN COMPANY.
Bombay and Calcutta: MACMILLAN AND CO., Ltd.

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THE THEORY OF DETERMINANTS

AND THEIR APPLICATIONS

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CAMBRIDGE
at the University Press
1904

Cambridge:
PRINTED BY J. AND C. F. CLAY,
AT THE UNIVERSITY PRESS.

PREFACE TO THE FIRST EDITION.

IN the present treatise I have attempted to give an exposition of the Theory of Determinants and their more important applications. In every case where it was possible I have consulted the original works and memoirs on the subject; a list of those I have been able to see is appended as it may be useful to others pursuing the same line of study. At one time I hoped to make this list exhaustive, supplementing my own researches from the literary notices in foreign mathematical journals, but even with this aid I found that it would be necessarily incomplete. In consequence of this the list has been restricted to those memoirs which I have seen, the leading results of which are incorporated either in the body of the text or in the examples.

The principal novelty of the treatise lies in the systematic use of Grassmann's alternate units, by means of which the study of determinants is, I believe, much simplified.

I have to thank my friend Mr JAS. BARNARD, M.A. of St John's College and Mathematical Master at the Proprietary School, Blackheath, for the care he has bestowed on correcting the proofs and for many valuable suggestions.

R. F. SCOTT.

February 1880.

REVISER'S PREFACE.

THE principal changes made in this edition are that some account has been given of infinite determinants, and of the elements of the theory of bilinear forms, together with the fundamental propositions about elementary divisors. I have intentionally refrained, as far as possible, from altering the character of the book, or increasing its size. The list of books and memoirs relating to determinants has been omitted, Dr Muir's bibliography being easily accessible; instead of this I have given a brief account of the earlier history of the subject. The new introductory chapter is intended for beginners, who are apt to feel discouraged if they first approach the theory in its most general form. For a similar reason the abbreviated notation employed in some chapters has not been used in those which are more elementary.

Besides original papers, I have consulted Pascal's excellent treatise in the Hoepli series, and Muth's *Elementartheiler*. The first volume of Kronecker's lectures on determinants appeared too late for me to consult it. I ought to say that for all the changes that have been made I am solely responsible, the revision having been left entirely in my hands by the author.

G. B. MATHEWS.

May 1904.

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THEORY OF DETERMINANTS.

CHAPTER I.

INTRODUCTION.

1. DETERMINANTS are algebraical expressions of a particular type calculated by a systematic rule and expressed by a special notation. Accordingly there is a calculus of determinants; and besides this there is a theory, dealing with the properties of determinants which result from their analytical form.

The most natural way of introducing the subject is to consider a few simple cases of the problem to which the invention of determinants is due; namely, the formal solution of a general system of simultaneous linear equations.

If the system is

$$\left.\begin{array}{l} a_1x + a_2y + a_3 = 0 \\ b_1x + b_2y + b_3 = 0 \end{array}\right\}$$

the solution is at once found to be

$$x = \frac{a_2b_3 - a_3b_2}{a_1b_2 - a_2b_1}, \quad y = \frac{a_3b_1 - a_1b_3}{a_1b_2 - a_2b_1};$$

and in the same way the homogeneous equations

$$\left.\begin{array}{l} a_1x + a_2y + a_3z = 0 \\ b_1x + b_2y + b_3z = 0 \end{array}\right\}$$

lead to the proportion

$$x : y : z = (a_2b_3 - a_3b_2) : (a_3b_1 - a_1b_3) : (a_1b_2 - a_2b_1),$$

from which the previous result follows by putting $z = 1$.

Consider next the homogeneous system

$$\left.\begin{aligned} u_1 &\equiv a_1x + a_2y + a_3z + a_4t = 0 \\ u_2 &\equiv b_1x + b_2y + b_3z + b_4t = 0 \\ u_3 &\equiv c_1x + c_2y + c_3z + c_4t = 0 \end{aligned}\right\}.$$

In the derived equation

$$\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 = 0$$

the coefficients of z and t will vanish if

$$\lambda_1 a_3 + \lambda_2 b_3 + \lambda_3 c_3 = 0,$$

$$\lambda_1 a_4 + \lambda_2 b_4 + \lambda_3 c_4 = 0;$$

that is, by the preceding case, if

$$\lambda_1 : \lambda_2 : \lambda_3 = (b_3c_4 - b_4c_3) : (c_3a_4 - c_4a_3) : (a_3b_4 - a_4b_3).$$

Taking the multipliers in this proportion, the derived equation becomes

$$Qx + Py = 0,$$

where

$$P = a_2(b_3c_4 - b_4c_3) + b_2(c_3a_4 - c_4a_3) + c_2(a_3b_4 - a_4b_3),$$

$$Q = a_1(b_3c_4 - b_4c_3) + b_1(c_3a_4 - c_4a_3) + c_1(a_3b_4 - a_4b_3).$$

The expressions P, Q are of precisely similar form, and differ only in the sets of coefficients which they involve. It is convenient to write

$$P = \begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix};$$

in this notation P is said to be expressed as a *determinant of the third order*. The determinant has three *rows*, such as a_2, a_3, a_4; three *columns*, such as a_2, b_2, c_2; and nine *elements*, a_2, a_3, ... c_4.

With the same notation

$$Q = \begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix};$$

and it is found without difficulty that the complete solution of the given system is

$$\frac{x}{\begin{vmatrix} a_2 & a_3 & a_4 \\ b_2 & b_3 & b_4 \\ c_2 & c_3 & c_4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & a_3 & a_4 \\ b_1 & b_3 & b_4 \\ c_1 & c_3 & c_4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & a_2 & a_4 \\ b_1 & b_2 & b_4 \\ c_1 & c_2 & c_4 \end{vmatrix}} = \frac{-t}{\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}}.$$

2. We may use this result to solve, by the method of multipliers, a system of four homogeneous equations in five unknowns, and so on. The general result, as will be proved later on, is that $(n+1)$ variables satisfying n linear homogeneous equations are proportional to $(n+1)$ rational integral functions of the coefficients, each homogeneous and of dimension n in a certain set of n^2 coefficients. These expressions are, in fact, determinants of the nth order, analogous to those of the third order above defined.

A determinant of the first order, $|a|$, is merely the single term a; for the second order, we have

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1;$$

the general rules for expanding a determinant of any order will be found in the next chapter.

3. A convenient rule for writing down the expansion of any determinant of the third order is the following, due to Sarrus.

Let the determinant be

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Alongside of this repeat the first and second columns in order

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2 \\ c_1 & c_2 & c_3 & c_1 & c_2 \end{array}$$

and form the product of each set of three elements lying in lines

parallel to the diagonals of the original square. Those which lie in lines descending from left to right have the positive, the others the negative sign. Thus the determinant is

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2$$
$$- a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3.$$

In practice it is not necessary actually to repeat the columns, but only to imagine them repeated.

4. In order to make the notation familiar, proofs will now be given of some elementary properties of determinants of the third order. They are special cases of theorems which are true for any order; and the reader will easily verify them for determinants of the second order.

Consider the determinant

$$D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

Each term of the expansion is of the form $\pm a_\alpha b_\beta c_\gamma$, where (α, β, γ) is a permutation of (1, 2, 3); in other words, it is a product of three elements, no two of which belong to the same row or to the same column. The sign of the term $a_1b_2c_3$, derived from the principal diagonal (that which slopes down from left to right), is positive: any other term $a_\alpha b_\beta c_\gamma$ is preceded by the sign $-$ or $+$ according as (α, β, γ) is derivable from (1, 2, 3) by a single transposition or two transpositions. Thus half the terms are positive, and half negative.

The value of D is unaltered if columns are changed into rows without altering the positions of a_1, b_2, c_3: that is to say,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

This is verified by expanding each side.

5. If any two rows or columns of D are interchanged, the value of the new determinant is $-D$. For the effect of the

change on the expansion of D is either to change two letters (such as a, b) without altering the positions of the suffixes, or else to interchange two suffixes: in each case the result is $-D$. For instance, if the second and third rows change places, the new determinant is

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_1c_2b_3 - a_1c_3b_2 + a_2c_3b_1 - a_2c_1b_3 + a_3c_1b_2 - a_3c_2b_1$$

$$= -D.$$

Hence if two rows or two columns are identical, the value of D is zero.

6. We have

$$D = a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix};$$

thus D is a linear homogeneous function of the elements a_1, a_2, a_3 in the first row, the coefficients being determinants of the second order constructed from elements in the other rows. D can be similarly expressed as a linear function of the elements of any other row or column.

It follows from this that

$$\begin{vmatrix} p_1+q_1, & p_2+q_2, & p_3+q_3 \\ b_1\ \ , & b_2\ \ , & b_3 \\ c_1\ \ , & c_2\ \ , & c_3 \end{vmatrix} = \begin{vmatrix} p_1 & p_2 & p_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} q_1 & q_2 & q_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix};$$

and there is a corresponding theorem when the elements of any column are binomials. More generally, if each element of the first column is expressed as the sum of l terms, each of the second as the sum of m terms, and each of the third as the sum of n terms, then D can be expressed as the sum of lmn determinants, in each of which the columns consist of corresponding terms of the sums in question. There is, of course, an analogous theorem with "rows" instead of "columns."

Again, if the elements of any row or column are multiplied by k, the value of D is multiplied by k. For instance,

$$\begin{vmatrix} ka_1 & a_2 & a_3 \\ kb_1 & b_2 & b_3 \\ kc_1 & c_2 & c_3 \end{vmatrix} = k \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

By combining the foregoing results, we obtain the useful theorem that

$$\begin{vmatrix} ka_1 + la_2 + ma_3, & a_2, & a_3 \\ kb_1 + lb_2 + mb_3, & b_2, & b_3 \\ kc_1 + lc_2 + mc_3, & c_2, & c_3 \end{vmatrix} = k \times \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = kD.$$

This follows from the fact that the determinant on the left can be expressed in the form $k\Delta_1 + l\Delta_2 + m\Delta_3$, where $\Delta_1 = D$, and the determinants Δ_2, Δ_3 vanish on account of the identity of two columns.

7. The product of two determinants of the third order can be expressed as a determinant of the third order.

To see this, consider the determinant

$$D = \begin{vmatrix} a\alpha + b\beta + c\gamma, & a\alpha' + b\beta' + c\gamma', & a\alpha'' + b\beta'' + c\gamma'' \\ a'\alpha + b'\beta + c'\gamma, & a'\alpha' + b'\beta' + c'\gamma', & a'\alpha'' + b'\beta'' + c'\gamma'' \\ a''\alpha + b''\beta + c''\gamma, & a''\alpha' + b''\beta' + c''\gamma', & a''\alpha'' + b''\beta'' + c''\gamma'' \end{vmatrix}.$$

By selecting partial columns, this can be expressed as the sum of 27 determinants such as

$$\begin{vmatrix} a\alpha & a\alpha' & b\beta'' \\ a'\alpha & a'\alpha' & b'\beta'' \\ a''\alpha & a''\alpha' & b''\beta'' \end{vmatrix}, \quad \begin{vmatrix} a\alpha & b\beta' & c\gamma'' \\ a'\alpha & b'\beta' & c'\gamma'' \\ a''\alpha & b''\beta' & c''\gamma'' \end{vmatrix},$$

and so on. But of these the only ones that do not vanish are the six which, like the second one above written, contain all the nine elements of the determinant

$$\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}.$$

Calling this d, and putting

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = \delta,$$

each of the set of 27 which does not vanish is the product of d by a term of δ; for instance, the one above given is equal to $\alpha\beta'\gamma'' d$. Moreover it is found, on examination, that the sign of the term which multiplies d agrees with the sign which it has in the expansion of δ; and that every term of δ is represented in this way. Consequently

$$D = d\delta.$$

8. In expressing the product $d\delta$ as above, the multiplication is said to be effected by rows; in fact each element of D is the sum of products of corresponding pairs of elements of rows in d and δ. Since the values of d, δ are not affected by interchanging rows and columns, there are four ways of performing the multiplication: the resulting determinants are, in general, distinct in form, though their complete expansions are, of course, the same identically.

To illustrate the different methods of procedure, we may take the case of two determinants of the second order. By the same kind of reasoning as before, it is verified that each of the determinants

$$\begin{vmatrix} a\alpha + b\beta, & a\alpha' + b\beta' \\ a'\alpha + b'\beta, & a'\alpha' + b'\beta' \end{vmatrix}, \quad \begin{vmatrix} a\alpha + b\alpha', & a\beta + b\beta' \\ a'\alpha + b'\alpha', & a'\beta + b'\beta' \end{vmatrix},$$

$$\begin{vmatrix} a\alpha + a'\beta, & a\alpha' + a'\beta' \\ b\alpha + b'\beta, & b\alpha' + b'\beta' \end{vmatrix}, \quad \begin{vmatrix} a\alpha + a'\alpha', & a\beta + a'\beta' \\ b\alpha + b'\alpha', & b\beta + b'\beta' \end{vmatrix}$$

is equal to the product of $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$ and $\begin{vmatrix} \alpha & \beta \\ \alpha' & \beta' \end{vmatrix}$.

CHAPTER II.

DEFINITIONS AND NOTATION. ALTERNATE NUMBERS.

1. Before proceeding to the theory of determinants of any order, it is convenient to recall a few theorems relating to the permutations of n different things in a line.

If we have any n elements $a_1, a_2, \dots a_n$, we may call

$$a_1, \ a_2, \ \dots \ a_n,$$

where the elements are arranged according to the magnitude of the numbers forming the suffixes, the natural or original order of the letters. Any other order is called a permutation of the elements. One element is said to be higher than another when it has the greater suffix. When in any permutation an element with a higher suffix precedes another with a lower, we have an *inversion*.

Thus the permutation a_4, a_2, a_1, a_3, of four letters, contains the following four inversions,

$$a_4a_2, \quad a_4a_1, \quad a_4a_3, \quad a_2a_1,$$

where we compare each element with all that follow it.

Following Cramer it is usual to divide the permutations of a given set of elements into two classes; the first class contains those permutations which have an even number of inversions, the second those which have an odd number.

2. By permuting the elements $a_1, a_2, \ldots a_n$ we obtain all possible ways in which they can be written. The same result is arrived at by writing down all the permutations of the suffixes $1, 2, \ldots n$ and then putting a's above them.

By repeated interchange of two suffixes we can get every permutation of the given elements from their original order.

For if we start with two suffixes 1, 2, they have but two arrangements,

$$1,\ 2, \qquad 2,\ 1,$$

of which the second is got from the first by a simple interchange. Taking three elements 1, 2, 3, out of these we can select the duad 2, 3, whose permutations are 2, 3; 3, 2. Prefixing 1 to each of these we get 1, 2, 3; 1, 3, 2, which are two permutations of the given elements. Proceeding in like manner with the other duads 1, 3; 1, 2, we get the six arrangements of three figures

$$\begin{array}{ccc} 1\ 2\ 3, & 1\ 3\ 2, & 2\ 3\ 1 \\ 2\ 1\ 3, & 3\ 1\ 2, & 3\ 2\ 1. \end{array}$$

Next take four numbers 1, 2, 3, 4. We get four triplets by leaving out one number, viz.

$$1\ 2\ 3, \qquad 1\ 2\ 4, \qquad 1\ 3\ 4, \qquad 2\ 3\ 4.$$

For each triplet we can write down six arrangements by the rule just given for three numbers, then adding on the missing number we get twenty-four arrangements of four numbers, viz.

$$\begin{array}{cccc} 1\ 2\ 3\ 4 & 1\ 2\ 4\ 3 & 1\ 3\ 4\ 2 & 2\ 3\ 4\ 1 \\ 2\ 1\ 3\ 4 & 2\ 1\ 4\ 3 & 3\ 1\ 4\ 2 & 3\ 2\ 4\ 1 \\ 1\ 3\ 2\ 4 & 1\ 4\ 2\ 3 & 1\ 4\ 3\ 2 & 2\ 4\ 3\ 1 \\ 3\ 1\ 2\ 4 & 4\ 1\ 2\ 3 & 4\ 1\ 3\ 2 & 4\ 2\ 3\ 1 \\ 2\ 3\ 1\ 4 & 2\ 4\ 1\ 3 & 3\ 4\ 1\ 2 & 3\ 4\ 2\ 1 \\ 3\ 2\ 1\ 4 & 4\ 2\ 1\ 3 & 4\ 3\ 1\ 2 & 4\ 3\ 2\ 1. \end{array}$$

And so we could go on to write down the arrangements of any set of elements.

The number of arrangements of n letters is $1 \cdot 2 \cdot 3 \ldots n$ or $n!$, an even number.

3. If in a given permutation two elements be interchanged while all the others remain unaltered in position, the two resulting permutations belong to different classes. This will be proved if we can shew that the difference between the number of inversions in the two permutations is an odd number.

We can represent any permutation of a group of elements by

$$A \ d \ B \ e \ C \qquad \text{.........................(1)},$$

where d and e are the two elements to be presently interchanged, A the group of elements which precede d, B the group between d and e, and C the group which follows e. The permutation we obtain is

$$A \ e \ B \ d \ C \qquad \text{.........................(2)}.$$

The number of inversions in the two permutations (1) and (2) due to the elements contained in the groups A, B and C is in each case the same. And since the elements of A precede d and e in both permutations we get no new inversions in (2) from these; the elements of C follow both d and e, and therefore give rise to no new inversions. We have therefore only to consider the changes in the two permutations

$$d \ B \ e \ \text{ and } \ e \ B \ d \qquad \text{.....................(3)}.$$

Suppose that e is higher than d; let B contain b elements of which b_1 are higher than d and b_2 higher than e. Then in the permutation $d \ B \ e$ we have, independently of the inversions contained in B itself, $b - b_1 + b_2$ inversions, because there are $b - b_1$ elements lower than d and b_2 higher than e.

In $e \ B \ d$ we have $b - b_2$ inversions on account of e, b_1 on account of d, and one because e is higher than d; thus, without counting the inversions in B, we have $b - b_2 + b_1 + 1$. The difference between the number of inversions in the permutations (3), and therefore in (1) and (2), is thus

$$b - b_2 + b_1 + 1 - (b - b_1 + b_2) = 2\,(b_1 - b_2) + 1,$$

which is an odd number, shewing that the permutations belong to different classes.

4. The same result may be arrived at as follows. If there be n quantities whose natural order is

$$a_1,\ a_2,\ \dots\ a_n,$$

and if in any arrangement we subtract each suffix from all that follow it and multiply these differences together, we shall have a product whose sign will depend on the number of inversions in the given arrangement, the sign being positive if the number of inversions is even and negative if the number of inversions is odd. If then i, k be any two suffixes chosen arbitrarily which are to be interchanged, i preceding k in the given arrangement, the product of the differences will consist of four parts.

(i) The factor $k-i$.

(ii) and (iii). A set of factors such as $\pm(r-k)$, and $\pm(r-i)$, where r is some number of the series $1 \dots n$ excluding i and k.

(iv) A set of factors such as $r-s$, where r, s are two numbers of the series $1, 2, \dots n$ excluding i and k.

Then for the given arrangement the product of the differences will be

$$\epsilon\,(k-i)\,\Pi\,(r-i)\,(r-k)\,\Pi\,(r-s),$$

where ϵ denotes $+1$ or -1 as the case may be. If now we interchange i and k, the signs of all factors such as $(r-k)(r-i)$, $(r-s)$ remain unchanged, while $k-i$ changes sign.

Thus on interchanging two elements the product of the differences changes sign, i.e. by interchanging two suffixes we have introduced an odd number of negative factors and therefore of inversions, hence the two arrangements considered belong to different classes.

5. If in a series of elements each is replaced by the one which follows it, and the last by the first, we are said to have got a cyclical permutation of the given arrangement. If the system of elements

$$a_1,\ a_2,\ \dots\ a_n$$

be considered as forming an endless band, if we cut this band between a_1 and a_n we have the natural order, cutting it between

a_1 and a_2 we have a cyclical permutation of the first order, and so on.

Such a cyclical permutation is equivalent to $n-1$ simple interchanges, viz. we move a_1 from the first to the last place by interchanging the first and second elements, then the second and third, and so on, in all $n-1$ simple interchanges. Thus a cyclical permutation of a given arrangement belongs to the same or opposite class as the given one according as the number of elements is odd or even.

6. Every permutation of a given set of elements may be considered as derived from a fixed permutation by means of cyclical permutations of groups of the elements.

This is best illustrated by an example. Let the suffixes of two permutations of nine elements be

$$\begin{array}{c} 7,\ 6,\ 3,\ 2,\ 1,\ 4,\ 8,\ 5,\ 9 \\ 8,\ 7,\ 9,\ 5,\ 1,\ 6,\ 4,\ 3,\ 2 \end{array}$$

To obtain the second permutation from the first, we begin by replacing 7 by 8, 8 by 4, 4 by 6 and 6 by 7, which completes a cycle. Then we replace 3 by 9, 9 by 2, 2 by 5 and 5 by 3, which completes another cycle. Lastly, 1 forms a cycle by itself.

7. If elements which remain unchanged like 1 in the preceding example be considered as forming a cycle of one letter, we may state the following theorem: Two permutations belong to the same or different classes, according as the difference between the number of elements and the number of groups by whose cyclical interchange one permutation is got from the other, is even or odd.

For if there be n elements altogether, and p cycles of n_1, $n_2 \dots n_p$ letters respectively, the cyclical interchanges are equivalent to

$$\begin{aligned}(n_1-1)+(n_2-1)+\dots+(n_p-1) &= n_1+n_2+\dots+n_p-p \\ &= n-p\end{aligned}$$

simple interchanges, which proves the theorem.

In the example in Art. 6, $n=9$, $p=3$, and thus the permutations belong to the same class.

8. A determinant of the nth order is a function of n^2 elements, which are conveniently distinguished by double suffixes in the following manner. The complete symbol for the determinant is written

$$\begin{vmatrix} a_{11} & a_{12} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \dots & a_{nn} \end{vmatrix},$$

so that the element a_{rs} is the sth constituent of the rth row, or (which is the same thing) the rth constituent of the sth column; rows being counted from above downwards, and columns from left to right. The elements may be regarded as being arranged in a square block; of this the line containing $a_{11}, a_{22}, \dots a_{nn}$ is called the leading or principal diagonal.

The actual block of elements, considered *per se*, is called an array, or matrix. More exactly, it is a square array, as distinguished from other arrays.

The expansion of the determinant represented by the above symbol is obtained by the following rule: From the array choose n different elements such that there is one and only one element from each row and column, and multiply these elements together; the product will be a term of the determinant. For example, the product of the elements

$$a_{11}, \; a_{22}, \dots a_{nn}$$

situated in the principal diagonal of the square array, is a term of the determinant; this will be called the leading term, and to it we attribute the positive sign.

The sign of any other term

$$a_{fg} \, . \, a_{hk} \dots a_{st}$$

is determined as follows. From the mode in which the elements were selected, it follows that

$$f, \; h, \dots s, \quad \text{and} \quad g, \; k, \dots t$$

are each of them permutations of $1, 2, \dots n$. Let them contain p and q inversions respectively, then the sign of the term

$$a_{fg} \, . \, a_{hk} \dots a_{st}$$

is $(-1)^{p+q}$. The sum of all the possible terms with their proper signs is the determinant of the array.

More simple rules may be given for determining the sign of any term. If we interchange any two elements a_{hk} and a_{ij} the term does not change its sign. For this interchange is equivalent to the interchange of i with h and j with k. By these two interchanges we increase both p and q by an odd number, and hence the sign of the term is unaltered. It is therefore usual to give to one series of suffixes their natural order, so that one of the two numbers p or q becomes zero, and the sign of the term of the determinant now depends solely on the number of inversions in the other series, and is the same whether the first or second series of suffixes retains its natural order.

It is thus clear that all the terms of the determinant will be obtained from the leading term

$$a_{11}a_{22}\ldots a_{nn}$$

by keeping the first suffixes fixed in their natural order, and writing for the second suffixes in succession all possible permutations of the elements 1, 2, ... n, giving to the product of the elements the positive or negative sign according as the number of inversions is even or odd.

Such a determinant is said to be of the nth degree, since each term is the product of n elements. It has $n!$ terms in all, since this is the number of permutations of the second suffixes, each of which gives a term of the determinant. Half of these terms have the positive, the rest the negative sign.

9. Various notations are employed for the determinant of a system of n^2 elements. Cauchy and Jacobi denoted it by drawing two vertical lines at the sides of the array, or by writing $\pm$ before the leading term and prefixing a summation sign,

$$\begin{vmatrix} a_{11}, & a_{12}, & \ldots & a_{1n} \\ a_{21}, & a_{22}, & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1}, & a_{n2}, & \ldots & a_{nn} \end{vmatrix} = \Sigma \pm a_{11}a_{22}\ldots a_{nn}.$$

Sylvester uses the umbral notation

$$\begin{vmatrix} 1, & 2, & \ldots & n, \\ 1, & 2, & \ldots & n. \end{vmatrix}$$

If the determinant be written in the form

$$\begin{vmatrix} x_1, & y_1, & z_1, & \ldots \\ x_2, & y_2, & z_2, & \ldots \\ \cdots & \cdots & \cdots & \cdots \\ x_n, & y_n, & z_n, & \ldots \end{vmatrix}$$

we may denote it by

$$|x_i,\ y_i,\ z_i,\ \ldots| \quad (i=1,\ 2,\ \ldots\ n),$$

meaning by this that i is to take the different values $1, 2, \ldots n$ in succession. Lastly, the determinant with double suffixes may be denoted by

$$|a_{ik}| \quad (i,\ k=1,\ 2,\ \ldots\ n),$$

the bracket at the side telling us what values the suffixes i and k take.

This notation is, I believe, due to H. J. S. Smith, who employs it in his report on the theory of numbers, *Brit. Ass. Rep.*, 1861, p. 504. It clashes with Weierstrass's use of the symbol $|z|$ to denote the absolute value of the complex quantity z; but this does not lead practically to any confusion.

If we use the symbol $|a_{nn}|$, the explanatory bracket becomes unnecessary, since the notation now indicates the order of the determinant, and it is, of course, understood that the elements unexpressed are of the type a_{rs}, with $r \leqslant n$, $s \leqslant n$.

10. Illustrations of the rule for expanding a determinant will be found in the introductory chapter. It may be observed that when the elements are represented by literal symbols with single suffixes, it is usual to write the factors of any term in the dictionary order of the letters: its sign is then determined by the class of the permutation of the suffixes. For instance,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = \begin{aligned} & a_1b_2c_3d_4 - a_2b_1c_3d_4 - a_1b_3c_2d_4 + a_3b_1c_2d_4 \\ & + a_2b_3c_1d_4 - a_3b_2c_1d_4 - a_1b_2c_4d_3 + a_2b_1c_4d_3 \\ & + a_1b_4c_2d_3 - a_4b_1c_2d_3 - a_2b_4c_1d_3 + a_4b_2c_1d_3 \\ & + a_1b_3c_4d_2 - a_3b_1c_4d_2 - a_1b_4c_3d_2 + a_4b_1c_3d_2 \\ & + a_3b_4c_1d_2 - a_4b_3c_1d_2 - a_2b_3c_4d_1 + a_3b_2c_4d_1 \\ & + a_2b_4c_3d_1 - a_4b_2c_3d_1 - a_3b_4c_2d_1 + a_4b_3c_2d_1. \end{aligned}$$

In the double-suffix notation, the same expansion is, in the same order,

$$| a_{44} | = a_{11}a_{22}a_{33}a_{44} - a_{21}a_{12}a_{33}a_{44} - a_{11}a_{32}a_{23}a_{44} + \text{etc.}$$

11. If we interchange rows and columns in the determinant of Art. 9, we get

$$\begin{vmatrix} a_{11}, & a_{21}, & \dots & a_{n1} \\ a_{12}, & a_{22}, & \dots & a_{n2} \\ \dots & \dots & \dots & \dots \\ a_{1n}, & a_{2n}, & \dots & a_{nn} \end{vmatrix}.$$

This is the same as the original determinant with the suffixes of each element interchanged. Its expansion is then obtained from that of the original determinant by interchanging in each term the suffixes of each element. That is to say, in the term $a_{11}a_{22} \dots a_{nn}$ we keep the second suffixes fixed in their natural order and write for the first suffixes all possible permutations of $1, 2, \dots n$. But the reasoning of Art. 8 shews that each term in the new determinant has the same sign as the corresponding one in the original determinant.

Thus a determinant remains unchanged in value when its rows and columns are interchanged.

Alternate Numbers.

12. The magnitudes with which we deal in ordinary or arithmetical algebra are subject, as regards their addition and multiplication, to the following principal laws:

(i) The associative law, which states that

$$(a + b) + c = a + (b + c) = a + b + c,$$

and that $$ab \,.\, c = a \,.\, bc = abc.$$

(ii) The commutative law, which states that

$$a + b = b + a,$$

$$ab = ba.$$

(iii) The distributive law, which states that

$$(b+c)\,a = ba + ca,$$
$$a\,(b+c) = ab + ac.$$

The researches of modern algebraists have led them to consider quantities for which one or more of these laws ceases to hold, or for which one or more of these laws assumes a different form.

Numbers, whether real or ideal, which follow the laws of arithmetical algebra will be called *scalar* quantities.

We shall find it useful to consider a class of numbers which have received the name of alternate numbers. These are determined by means of a system of independent units given in sets like the co-ordinates of a point in space; such a set will be denoted by $e_1, e_2, \ldots e_n$. A number such as

$$A = a_1e_1 + a_2e_2 + \ldots + a_ne_n,$$

formed by adding the units together, each multiplied by a scalar, will be called an alternate number of the nth order.

In combination with scalar quantities and with units of other sets these units follow the laws of ordinary algebra. In combination with each other the units of a system follow the associative law and the commutative law as regards addition, but for multiplication we have the new equation

$$e_re_s = -e_se_r \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1),$$

when r, s are unequal; and

$$e_r^2 = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

for all values of r.

13. If $$A = a_1e_1 + a_2e_2 + \ldots + a_ne_n,$$
$$B = b_1e_1 + b_2e_2 + \ldots + b_ne_n$$

be two alternate numbers of the nth order, we define their product as follows:

$$\begin{aligned} AB &= \sum_r a_re_r \sum_s b_se_s \\ &= \sum_{r,s} a_re_r \,.\, b_se_s \\ &= \sum_{r,s} a_rb_se_re_s. \end{aligned}$$

Hence, by equations (1) and (2) of Art. 12,

$$AB = (a_1b_2 - a_2b_1)\,e_1e_2 + (a_1b_3 - a_3b_1)\,e_1e_3 + \dots + (a_{n-1}b_n - a_nb_{n-1})\,e_{n-1}e_n.$$

Thus clearly $AB = -BA$ and $A^2 = 0$, proving that alternate numbers have the same commutative law of multiplication as the units.

14. If k be any scalar

$$(A + kB)\,B = AB + kB^2 = AB,$$

so that the product of two alternate numbers is not altered if one be increased by a multiple of the other.

If we have a product of more than two numbers

$$ABC \dots\dots L,$$

it follows that for one of them, say C, we can write

$$C + k_1A + k_2B + \dots + k_rL,$$

and the product will still remain unaltered.

Alternate numbers belong to that class of algebraical magnitudes for which multiplication is a determinate, but division an indeterminate process. In fact

$$\frac{AB}{B} = A + kB,$$

where k is an arbitrary scalar.

The continued product $e_1e_2 \dots e_n$ of all the units of a set will in future be assumed to be unity. An explanation of this assumption will be given later on.

15. If, now, we take a square array of elements such as that in Art. 8, we can form a system of n alternate numbers of the nth order by taking the elements of each row to form the coefficients of the units in the numbers. Let P be the product of all these numbers, so that

$$P = (a_{11}e_1 + a_{12}e_2 + \dots + a_{1n}e_n)\,(a_{21}e_1 + a_{22}e_2 + \dots + a_{2n}e_n) \dots (a_{n1}e_1 + a_{n2}e_2 + \dots + a_{nn}e_n).$$

On multiplying out the factors on the right,

$$P = \Sigma a_{1p}a_{2q} \dots a_{ns}e_pe_q \dots e_s.$$

Since $e_p e_q \dots e_s = 0$ if any two units are alike, it follows that in every term on the right which does not vanish $p, q \dots s$ is a permutation of $1, 2 \dots n$. It follows at once from the law of multiplication (equation (1), Art. 12) that

$$e_p e_q \dots e_s = (-1)^\nu e_1 e_2 \dots e_n,$$

where ν is the number of inversions in the series $e_p e_q \dots e_s$.

Thus $$P = e_1 e_2 \dots e_n \Sigma (-1)^\nu a_{1p} a_{2q} \dots a_{ns},$$

but the term under the summation sign is a term of the determinant of the system of elements, with its proper sign. Thus

$$\begin{aligned} P &= |a_{nn}|\, e_1 e_2 \dots e_n \\ &= |a_{nn}|. \end{aligned}$$

Hence the determinant of a system of n^2 elements is expressed as a product of n alternate numbers linear in these elements. From this it immediately follows that if all the elements of a row are multiplied by the same number the determinant is multiplied by that number, and if all the elements of a row vanish the determinant vanishes.

In future we shall write for a determinant of the nth order whichever of the forms

$$|a_{nn}|, \quad \Pi A_r, \quad \Sigma \pm a_{11} a_{22} \dots a_{nn},$$

$(A_r = a_{r1} e_1 + a_{r2} e_2 + \dots + a_{rn} e_n)$ is most convenient.

16. If the determinant is so constituted that the different factors of which it is composed do not contain all the units, its evaluation is frequently effected with ease.

For example, the determinant

$$\begin{vmatrix} a_{11}, & 0\,, & 0 & \dots\dots & 0 \\ a_{21}, & a_{22}, & 0 & \dots\dots & 0 \\ a_{31}, & a_{32}, & a_{33} & \dots\dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & a_{n3} & \dots\dots & a_{nn} \end{vmatrix}$$

in which all the elements above the leading diagonal vanish reduces to the product $a_{11} a_{22} \dots a_{nn}$.

For it is equal to the product of the alternate numbers

$$
\begin{array}{l}
a_{11}e_1 \\
a_{21}e_1 + a_{22}e_2 \\
a_{31}e_1 + a_{32}e_2 + a_{33}e_3 \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{n1}e_1 + a_{n2}e_2 + a_{n3}e_3 + \ldots + a_{nn}e_n.
\end{array}
$$

Since the first number contains e_1, and e_1 only, all terms in the product of the remaining factors which contain e_1 disappear when multiplied by this factor, so that as far as we are concerned we may suppose $a_{21}, a_{31} \ldots a_{n1}$ to vanish. The second number reduces to $a_{22}e_2$, and the product of the first two to $a_{11}e_1a_{22}e_2$. We may shew in like manner that $a_{32}, a_{42} \ldots$ may vanish, and so on. Finally the product reduces to

$$a_{11}e_1a_{22}e_2 \ldots a_{nn}e_n = a_{11}a_{22} \ldots a_{nn}.$$

By an interchange of rows and columns it follows that the determinant for which all the elements below the leading diagonal vanish also reduces to its leading term.

17. As another example let us consider the determinant

$$
D = \begin{vmatrix}
0, & \cos(a_1 + a_2), & \cos(a_1 + a_3) \ldots\ldots \\
\cos(a_2 + a_1), & 0, & \cos(a_2 + a_3) \ldots\ldots \\
\cos(a_3 + a_1), & \cos(a_3 + a_2), & 0 \qquad \ldots\ldots \\
\cdots\cdots\cdots & \cdots\cdots\cdots & \cdots\cdots\cdots
\end{vmatrix}
$$

of order n: the element in the rth row and sth column is $\cos(a_r + a_s)$ unless $r = s$, when it vanishes.

Substitute for the cosines their exponential values and write

$$e^{a\sqrt{-1}} = \alpha.$$

Then D is the product of such factors as

$$\tfrac{1}{2}\left[\left(\alpha_1\alpha_2 + \frac{1}{\alpha_1\alpha_2}\right)e_2 + \left(\alpha_1\alpha_3 + \frac{1}{\alpha_1\alpha_3}\right)e_3 + \ldots + \left(\alpha_1\alpha_n + \frac{1}{\alpha_1\alpha_n}\right)e_n\right]$$

$$= \tfrac{1}{2}\left[\alpha_1 E + \frac{F}{\alpha_1} - \left(\alpha_1^2 + \frac{1}{\alpha_1^2}\right)e_1\right],$$

where
$$E = \alpha_1e_1 + \alpha_2e_2 + \ldots + \alpha_ne_n,$$

$$F = \frac{e_1}{\alpha_1} + \frac{e_2}{\alpha_2} + \ldots + \frac{e_n}{\alpha_n}.$$

Thus if $$\alpha_s E + \frac{F}{\alpha_s} = A_s,$$

we see that $$(-2)^n D = \Pi\,(2e_s \cos 2a_s - A_s).$$

Now observe that since the quantities A_s depend only on the two alternate numbers E and F, the product of more than two of them must vanish. Hence expanding

$$(-2)^n D = 2^n \cos 2a_1 \cos 2a_2 \ldots \cos 2a_n - 2^n \cos 2a_1 \ldots \cos 2a_n\, \Sigma \frac{e_1 e_2 \ldots A_n}{2\cos 2a_n}$$
$$+ 2^n \cos 2a_1 \ldots \cos 2a_n\, \Sigma \frac{e_1 e_2 \ldots e_{n-2} A_{n-1} A_n}{4 \cos 2a_{n-1} \cos 2a_n}.$$

Now $$e_1 e_2 \ldots e_{n-1} A_n = e_1 \ldots e_{n-1} \left(\alpha_n E + \frac{F}{\alpha_n}\right)$$
$$= \frac{1}{\alpha_n^2} + \alpha_n^2 = 2 \cos 2a_n.$$

$$e_1 e_2 \ldots A_{n-1} A_n = e_1 \ldots e_{n-2} \left(\frac{\alpha_{n-1}}{\alpha_n} - \frac{\alpha_n}{\alpha_{n-1}}\right) EF$$
$$= \left(\frac{\alpha_{n-1}}{\alpha_n} - \frac{\alpha_n}{\alpha_{n-1}}\right)^2 = -4 \sin^2 (a_{n-1} - a_n).$$

Thus $$\frac{(-1)^n D}{\cos 2a_1 \cos 2a_2 \ldots \cos 2a_n} = 1 - n - \Sigma \frac{\sin^2 (a_r - a_s)}{\cos 2a_r \cos 2a_s},$$

or $$\frac{(-1)^{n-1} D}{\cos 2a_1 \ldots \cos 2a_n} = (n-1) + \Sigma \frac{\sin^2 (a_r - a_s)}{\cos 2a_r \cos 2a_s},$$

where (r, s) are all duads derived from $1, 2 \ldots n$.

CHAPTER III.

GENERAL PROPERTIES OF DETERMINANTS.

1. If two columns or rows of a determinant be interchanged the resulting determinant is equal in value to the original, but of opposite sign.

Let $D = \Pi\,(a_{r1}e_1 + \ldots + a_{rs}e_s + \ldots + a_{rt}e_t + \ldots + a_{rn}e_n) = \Pi A_r$;
then, if D' is the determinant got by interchanging the sth and tth columns,

$$D' = \Pi\,(a_{r1}e_1 + \ldots + a_{rt}e_s + \ldots + a_{rs}e_t + \ldots + a_{rn}e_n);$$

but since in addition we follow the ordinary commutative law, D' is got from D by interchanging e_s and e_t in the product on the right. This leaves the scalar factor unaltered but changes the sign of the product of the units, thus

$$D' = -D.$$

Interchanging two rows of a determinant, say the rth and sth, is the same as interchanging the two factors A_r and A_s on the right: this is equivalent to an odd number of inversions, and hence by the rule of multiplication changes the sign of the product. This second argument, in fact, proves both parts of the proposition, since D is unaltered by changing rows into columns (II. 11).

2. If two rows or columns of a determinant be identical the determinant vanishes. For by the interchange of the two columns in question the determinant changes sign, but both columns being alike the determinant remains the same, thus

$$D = -D \text{ or } D = 0.$$

3. If each element of the rth row consist of the sum of two or more numbers the determinant splits up into the sum of two or more determinants having for elements of the rth row the separate terms of the elements of the rth row of the given determinant.

For if
$$D = \Pi A_s,$$
and
$$\begin{aligned} A_r &= (a_{r1} + b_{r1})\, e_1 + (a_{r2} + b_{r2})\, e_2 + \dots + (a_{rn} + b_{rn})\, e_n \\ &= (a_{r1}e_1 + \dots + a_{rn}e_n) + (b_{r1}e_1 + \dots + b_{rn}e_n) \\ &= A'_r + B_r; \end{aligned}$$
since
$$\begin{aligned} A_1 \dots A_r \dots A_n &= A_1 \dots (A'_r + B_r) \dots A_n \\ &= A_1 \dots A'_r \dots A_n + A_1 \dots B_r \dots A_n, \end{aligned}$$
we have
$$D = D_1 + D_2,$$
where D_1 and D_2 are determinants having for elements of the rth row in the sth place a_{rs} and b_{rs} respectively.

Repeated applications of this reasoning shew that if the elements of the rth row consist each of the sum of p elements, then the original determinant can be resolved into the sum of p determinants having for their rth rows the terms of the elements of the rth row of the given determinant.

The same theorem would apply if the elements of a column consisted of the sum of elements. In fact whenever a theorem applies to rows it applies equally to columns, as these can be interchanged (II. 11).

In future, when a theorem is stated with regard either to rows or to columns, it is to be understood as applying also to the other.

4. The value of a determinant is not altered if we add to the elements of any row the corresponding elements of another row, each multiplied by the same constant factor.

For if we add to the elements of the rth row those of the sth row, each multiplied by p, the resulting determinant is
$$\begin{aligned} A_1 \dots (A_r + pA_s) \dots A_s \dots A_n &= A_1 \dots A_r \dots A_s \dots A_n + pA_1 \dots A_s \dots A_s \dots A_n \\ &= A_1 \dots A_r \dots A_s \dots A_n, \end{aligned}$$
the other product vanishing, since it contains two identical factors.

For brevity the operation of adding corresponding elements of two rows is usually spoken of as adding the rows.

5. The theorem of the last article is of great importance in the reduction of determinants. The following are examples of its application:

(i) If corresponding elements of two rows of a determinant have a constant ratio the determinant vanishes. For we have only to multiply the elements of one row by a proper factor and subtract them from the elements of the other when all the elements in that row will vanish, and consequently the determinant vanishes.

Of a similar nature are the two following theorems, which may be proved without difficulty:

(ii) If the ratio of the differences of corresponding elements in the pth and qth rows to the difference of corresponding elements in the rth and sth rows be constant, then the determinant vanishes.

(iii) If from the corresponding elements of $l+1$ rows we form the lth differences and from the corresponding elements of $m+1$ rows the mth differences (the second set of rows being at least partially different from the first set); then, if the ratio of corresponding differences is constant, the determinant vanishes.

(iv) Let
$$D = \begin{vmatrix} u_1, & v_1 & \dots & t_1 \\ u_2, & v_2 & \dots & t_2 \\ \dots & \dots & \dots & \dots \\ u_n, & v_n & \dots & t_n \end{vmatrix}.$$

Subtract each row from the one which follows it, beginning with the last but one. Then, if

$$\Delta u_i = u_{i+1} - u_i,$$

we have
$$D = \begin{vmatrix} u_1, & v_1 & \dots & t_1 \\ \Delta u_1, & \Delta v_1 & \dots & \Delta t_1 \\ \Delta u_2, & \Delta v_2 & \dots & \Delta t_2 \\ \dots & \dots & \dots & \dots \\ \Delta u_{n-1}, & \Delta v_{n-1} & \dots & \Delta t_{n-1} \end{vmatrix}.$$

Repeat the same operation, stopping short at the second row.

Then, if $$\Delta^2 u_i = \Delta u_{i+1} - \Delta u_i,$$

$$D = \begin{vmatrix} u_1, & v_1 & \dots & t_1 \\ \Delta u_1, & \Delta v_1 & \dots & \Delta t_1 \\ \Delta^2 u_1, & \Delta^2 v_1 & \dots & \Delta^2 t_1 \\ \dots & \dots & \dots & \dots \\ \Delta^2 u_{n-2}, & \Delta^2 v_{n-2} & \dots & \Delta^2 t_{n-2} \end{vmatrix}.$$

Proceed in this way, leaving out a row each time, and we see that

$$D = \begin{vmatrix} u_1, & v_1 & \dots & t_1 \\ \Delta u_1, & \Delta v_1 & \dots & \Delta t_1 \\ \Delta^2 u_1, & \Delta^2 v_1 & \dots & \Delta^2 t_1 \\ \dots & \dots & \dots & \dots \\ \Delta^{n-1} u_1, & \Delta^{n-1} v_1 & \dots & \Delta^{n-1} t_1 \end{vmatrix};$$

where generally: $$\Delta^r u_i = \Delta^{r-1} u_{i+1} - \Delta^{r-1} u_i.$$

Suppose now that u_x is a polynomial in x of degree 0, v_x one of degree 1, and so on, then all the elements below the leading diagonal of D vanish, and

$$D = u_1 \,.\, \Delta v_1 \,.\, \Delta^2 w_1 \dots \Delta^{n-1} t_1.$$

For example, if

$$m_p = \frac{m(m-1)\dots(m-p+1)}{1\,.\,2\dots p}, \quad m_0 = 1,$$

$$\begin{vmatrix} m_0, & m_1 & \dots & m_r \\ (m+d)_0, & (m+d)_1 & \dots & (m+d)_r \\ \dots & \dots & \dots & \dots \\ (m+rd)_0, & (m+rd)_1 & \dots & (m+rd)_r \end{vmatrix} = 1\,.\,d\,.\,d^2 \dots d^r = d^{r(r+1)/2}.$$

For here $$\Delta^t (m+td)_t = d^t.$$

6. In a determinant of the form

$$\begin{vmatrix} 0, & 1, & 1, & 1 & \dots \\ 1, & a_{11}, & a_{12}, & a_{13} & \dots \\ 1, & a_{21}, & a_{22}, & a_{23} & \dots \\ 1, & a_{31}, & a_{32}, & a_{33} & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

every element of which a_{rs} is a type can be replaced by

$$A_{rs} = a_{rs} + h_r + k_s,$$

where h_r and k_s are arbitrary quantities, without altering the value of the determinant.

For multiply the first row by h_r and add it to the rth row, then in this row the first element is still 1, while in place of a_{rs} we have $a_{rs}+h_r$. Now multiply the first column by k_s and add it to the sth column; the element in the first row is still unchanged, while the element under discussion has become $a_{rs}+h_r+k_s$.

These transformations have left the value of the determinant unaltered.

7. We are now in a position to solve the system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= u_1, \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= u_2, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n &= u_n. \end{aligned}$$

Take alternate units $e_1, e_2 \dots e_n$; multiply the first equation by e_1, the second by $e_2, \dots$ the last by e_n, and add the results. We thus get

$$A_1x_1 + A_2x_2 + \dots A_nx_n = U,$$

where $A_r = \sum_s a_{sr}e_s$, and $U = \Sigma u_s e_s$.

Multiplying by $A_2A_3 \dots A_n$, we obtain

$$x_1 |a_{nn}| = \begin{vmatrix} u_1, & a_{12} \dots a_{1n} \\ u_2, & a_{22} \dots a_{2n} \\ \dots & \dots\dots \\ u_n, & a_{n2} \dots a_{nn} \end{vmatrix},$$

and in general x_s is obtained by substituting in the determinant $|a_{nn}|$ for the elements of the sth column the quantities $u_1, u_2, \dots u_n$, and dividing the resulting determinant by $|a_{nn}|$.

A system of $(n+1)$ homogeneous equations in n variables may be treated in the same way. We shall return to the subject of linear equations later on. p. 130.

8. If p rows of a determinant whose elements are rational integral functions of x become identical when $x=a$, then the determinant is divisible by $(x-a)^{p-1}$. For subtract any one of these rows from each of the remaining $p-1$ rows; the determinant remains unchanged, but now when $x=a$ all the elements of these $p-1$ new rows vanish, hence each element divides by $x-a$, and

thus dividing each of the $p-1$ rows by this factor we see that the determinant divides by $(x-a)^{p-1}$.

If when $x=a$ the rows are not equal, but only proportional, the theorem is still true.

Ex. The value of the determinant

$$\begin{vmatrix} x, & a \ldots\ldots a \\ a, & x \ldots\ldots a \\ \ldots & \ldots\ldots\ldots \\ a, & a \ldots\ldots x \end{vmatrix} \ (n \text{ rows})$$

is

$$\{x+(n-1)\,a\}\,(x-a)^{n-1}.$$

For if $x=a$ the n rows all become identical, thus the determinant divides by $(x-a)^{n-1}$.

Adding all the rows to the first, each element in that row becomes $x+(n-1)\,a$, this is therefore a factor in the determinant. Thus the determinant divides by

$$\{x+(n-1)\,a\}\,(x-a)^{n-1}.$$

This is of the same degree as the determinant, and as the coefficient of x^n in the determinant and in the product is unity the determinant must be equal to the product.

CHAPTER IV.

ON THE MINORS AND ON THE EXPANSION OF A DETERMINANT.

1. If from the n rows of the array

$$\begin{array}{l} a_{11},\ a_{12}\ \ldots\ a_{1n} \\ a_{21},\ a_{22}\ \ldots\ a_{2n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1},\ a_{n2}\ \ldots\ a_{nn} \end{array}$$

we select any p rows, and then from the new array which these form select p columns, these when written in the form of a determinant constitute a minor of the given system. Such a minor is said to be of the pth order.

Since we can select p rows from n in

$$\frac{n(n-1)\ldots(n-p+1)}{1.2\ldots p} = n_p$$

ways, and p columns from n columns in a like number of ways, it follows that the given system of order n has $(n_p)^2$ minors of order p.

2. If out of the $n-p$ rows which remain after the above p have been selected we take those $n-p$ columns whose column suffixes are different from those selected in the minor of order p, we have another determinant, of order $n-p$, said to be complementary to that of order p.

For example, in the determinant

$$\begin{vmatrix} a_{11}, & a_{12}, & a_{13}, & a_{14}, & a_{15} \\ a_{21}, & a_{22}, & a_{23}, & \dots & a_{25} \\ \dots & \dots & \dots & \dots & \dots \\ a_{51}, & a_{52}, & \dots & \dots & a_{55} \end{vmatrix},$$

$$\begin{vmatrix} a_{11}, & a_{12} \\ a_{21}, & a_{22} \end{vmatrix} \text{ and } \begin{vmatrix} a_{33}, & a_{34}, & a_{35} \\ a_{43}, & a_{44}, & a_{45} \\ a_{53}, & a_{54}, & a_{55} \end{vmatrix}$$

are complementary minors.

3. If $p = 1$, i.e. if we take a single element, the complementary minor is a determinant of order $n-1$, which is called the complement of the element. This complement is obtained from the original determinant by omitting the row and column in which the selected element stands. For example, the complement of the element a_{rs}, which we denote by A_{rs}, is

$$\begin{vmatrix} a_{1,1} & \dots & a_{1,s-1}, & a_{1,s+1} & \dots & a_{1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{r-1,1} & \dots & a_{r-1,s-1}, & a_{r-1,s+1} & \dots & a_{r-1,n} \\ a_{r+1,1} & \dots & a_{r+1,s-1}, & a_{r+1,s+1} & \dots & a_{r+1,n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{n,1} & \dots & a_{n,s-1}, & a_{n,s+1} & \dots & a_{n,n} \end{vmatrix}.$$

This is sometimes spoken of as a first minor of the given determinant. In like manner the determinant formed by omitting p rows and p columns would be called a pth minor: it is to be observed that a pth minor is a determinant of order $n-p$.

4. We may extend the meaning of complementary minors as follows: From the array in Art. 1 select p rows and p columns, then from those that remain q rows and q columns, from those that remain r rows and r columns, and so on. With the elements in these selected rows and columns form determinants; these will form a complementary system of minors if

$$p + q + r + \dots = n.$$

The number of ways in which we can form such a system is

$$\left\{\frac{n!}{p!\,q!\,r!\dots}\right\}^2.$$

It is of course permissible that one or more of the numbers $p, q, r \ldots$ should be unity; the corresponding minor is then a single element. For the determinant

$$\begin{vmatrix} a_{11} \ldots a_{16} \\ \ldots\ldots\ldots\ldots \\ a_{61} \ldots a_{66} \end{vmatrix}$$

the minors

$$\begin{vmatrix} a_{24}, & a_{25} \\ a_{34}, & a_{35} \end{vmatrix}; \quad \begin{vmatrix} a_{12}, & a_{13}, & a_{16} \\ a_{42}, & a_{43}, & a_{46} \\ a_{62}, & a_{63}, & a_{66} \end{vmatrix}; \quad a_{51}$$

form such a complementary system, and there are 3600 systems of this type.

5. We have hitherto only considered the product of a set of alternate numbers equal in number to the number of units. Let us now consider the product

$$(a_{11}e_1 + a_{12}e_2 + \ldots + a_{1n}e_n) \ldots (a_{m1}e_1 + a_{m2}e_2 + \ldots + a_{mn}e_n);$$

this is equal to

$$\Sigma a_{1p}a_{2q} \ldots a_{mr}e_pe_q \ldots e_r,$$

where $p, q \ldots r$ consist of all combinations m at a time from $1, 2 \ldots n$, repetitions being allowed.

First, if $m > n$, we must have repetitions in every term of the sum, and hence [II. 12, equation (2)] the whole vanishes.

If $m = n$, we have the case of II. 15, and the sum is the determinant $|a_{nn}|$.

But if $m < n$, the sum is formed by taking for $p, q \ldots r$ all m-ads from $1, 2 \ldots n$ and permuting the elements of each m-ad in all possible ways.

Namely, the term

$$a_{1p}a_{2q} \ldots a_{mr}e_pe_q \ldots e_r$$

is got by taking $a_{1p}e_p$ from the first factor of the product, $a_{2q}e_q$ from the second ..., and $a_{mr}e_r$ from the last factor. But we should still get the product of the units $e_pe_q \ldots e_r$, though in a different order, if we take the pth term of some other factor than the first, the qth

of some other than the second, and so on. The term of the product which multiplies $e_p e_q \dots e_r$ is thus got from

$$a_{1p} a_{2q} \dots a_{mr}$$

by permuting $p, q \dots r$ in all possible ways, and giving to each term the sign corresponding to the number of inversions in its second suffixes, $p, q \dots r$ being considered the original order. The sum of these products is

$$\begin{vmatrix} a_{1p}, & a_{1q} & \dots & a_{1r} \\ a_{2p}, & a_{2q} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{mp}, & a_{mq} & \dots & a_{mr} \end{vmatrix}.$$

Hence the product of the m factors is equal to

$$\Sigma \begin{vmatrix} a_{1p}, & a_{1q} & \dots & a_{1r} \\ a_{2p}, & a_{2q} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{mp}, & a_{mq} & \dots & a_{mr} \end{vmatrix} e_p e_q \dots e_r \dots\dots\dots\dots\dots\dots(1).$$

In like manner, if we take the remaining factors necessary to form the determinant $|a_{nn}|$, we have

$$(a_{m+1,1} e_1 + \dots + a_{m+1,n} e_n) \dots (a_{n,1} e_1 + \dots + a_{n,n} e_n)$$
$$= \Sigma \begin{vmatrix} a_{m+1,u}, & a_{m+1,v} & \dots & a_{m+1,w} \\ a_{m+2,u}, & a_{m+2,v} & \dots & a_{m+2,w} \\ \dots & \dots & \dots & \dots \\ a_{n,u}, & a_{n,v} & \dots & a_{n,w} \end{vmatrix} e_u e_v \dots e_w \dots\dots\dots\dots(2),$$

where $u, v \dots w$ is a combination of $n-m$ numbers selected from $1, 2 \dots n$.

Now multiply the equation (2) by the equation (1) and we obtain

$$|a_{nn}| = \Sigma \left\{(-1)^\nu \begin{vmatrix} a_{1p}, & a_{1q} & \dots & a_{1r} \\ \dots & \dots & \dots & \dots \\ a_{mp}, & a_{mq} & \dots & a_{mr} \end{vmatrix} \begin{vmatrix} a_{m+1,u} & \dots & a_{m+1,w} \\ \dots & \dots & \dots \\ a_{n,u} & \dots & a_{n,w} \end{vmatrix} \right\};$$

where from the nature of the alternate numbers e it follows that the two determinant factors under the summation sign are complementary minors, and ν is the number of inversions in

$$e_p e_q \dots e_r e_u e_v \dots e_w \text{ or in } p, q \dots r, u, v \dots w.$$

This theorem, usually called Laplace's theorem, gives the expansion of a determinant in the form of a sum of products of complementary minors.

It is assumed in the above that the complementary minors are formed from the first m and last $n-m$ rows. Since by a suitable change of the order of the rows and sign of the determinant any m rows can be brought into the first m places, this is no real restriction.

6. For example, we have

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ \hline c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = \begin{array}{l} (12)(34)+(23)(14)+(31)(24) \\ +(34)(12)+(14)(23)+(24)(31), \end{array}$$

where for brevity

$$(12)(34) = \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \cdot \begin{vmatrix} c_3, & c_4 \\ d_3, & d_4 \end{vmatrix} \text{ \&c.}$$

In like manner

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4, & a_5 \\ b_1, & b_2, & b_3, & b_4, & b_5 \\ c_1, & c_2 \ldots\ldots & & & \\ \hline d_1, & d_2 \ldots\ldots & & & \\ e_1, & e_2 \ldots\ldots & & & \end{vmatrix} = \begin{array}{l} (123)(45)+(142)(35)+(134)(25)+(243)(15) \\ +(125)(34)+(315)(24)+(235)(14) \\ +(145)(23)+(425)(13) \\ +(345)(12), \end{array}$$

where

$$(123)(45) = \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} \cdot \begin{vmatrix} d_4, & d_5 \\ e_4, & e_5 \end{vmatrix} \text{ \&c.}$$

7. If when the determinant is divided into two sets of m and $n-m$ rows there are $n-m$ columns of zeros in the set of m rows, the determinant reduces to the product of the minor of the remaining m columns and its complementary minor.

This is clear, for with the exception of this single minor of order m all the others vanish because they contain at least one column of zero elements.

If the set of m rows contains more than $n-m$ columns of zeros the determinant vanishes.

Thus, for example:

$$\begin{vmatrix} a_1, & a_2, & 0, & 0 \\ b_1, & b_2, & 0, & 0 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = \begin{vmatrix} a_1, & a_2 \\ b_1, & b_2 \end{vmatrix} \begin{vmatrix} c_3, & c_4 \\ d_3, & d_4 \end{vmatrix},$$

while

$$\begin{vmatrix} a_1, & a_2, & 0, & 0, & 0 \\ b_1, & b_2, & 0, & 0, & 0 \\ c_1, & c_2, & 0, & 0, & 0 \\ d_1, & d_2, & d_3, & d_4, & d_5 \\ e_1, & e_2, & e_3, & e_4, & e_5 \end{vmatrix} = 0.$$

8. In Art. 5 we resolved a determinant into the sum of products of pairs of complementary minors. We can however resolve it into a sum of products of as many complementary minors as we please.

For we can divide up the n factors whose product is $|a_{nn}|$ as follows: Take the first u, the second $v \dots$, the last w. The product of the first u factors would be of the form

$$\Sigma \begin{vmatrix} a_{1p}, & a_{1q} & \dots & a_{1r} \\ a_{2p}, & a_{2q} & \dots & a_{2r} \\ \dots & \dots & \dots & \dots \\ a_{up}, & a_{uq} & \dots & a_{ur} \end{vmatrix} e_p e_q \dots e_r,$$

or

$$\Sigma D_u e_p e_q \dots e_r,$$

$p, q \dots r$ being u numbers taken from $1, 2 \dots n$ without repetition and D_u a minor of order u from the first u rows.

In like manner the product of the next v factors would be

$$\Sigma D_v e_f e_g \dots e_h,$$

D_v being a minor of order v chosen from the v rows.

Lastly, the product of the w factors would be

$$\Sigma D_w e_r e_s \dots e_t,$$

with a similar meaning for the quantities involved.

Now form the product of all the factors, taking care to keep them in their proper order; then

$$|a_{nn}| = \Sigma D_u D_v \dots D_w,$$

where $D_u, D_v, \dots D_w$ form a system of complementary minors of the determinant $|a_{nn}|$.

The sign of the term is determined from the number of inversions in

$$p,\ q \dots r,\ f,\ g \dots h,\ r,\ s \dots t.$$

9. If in Art. 5 we restrict the first product to the single factor

$$a_{r1}e_1 + a_{r2}e_2 + \dots + a_{rn}e_n \dots\dots\dots\dots\dots\dots (1),$$

the second product becomes

$$A_{r1}E_1 + A_{r2}E_2 + \dots + A_{rn}E_n \dots\dots\dots\dots\dots (2),$$

where A_{rs} is the complement of a_{rs} (Art. 3) and

$$E_s = e_1 e_2 \dots e_{s-1} e_{s+1} \dots e_n.$$

For we get a term of the product by leaving out each unit such as e_s in turn, i.e. by forming a determinant with the remaining $n-1$ columns; and since we previously omitted the rth row of the given determinant, this determinant is A_{rs}.

Now multiply the $n-1$ factors which form (2) by the remaining factor (1); we obtain

$$(-1)^{r-1}|a_{nn}| = a_{r1}A_{r1} - a_{r2}A_{r2} + \dots + (-1)^{s-1} a_{rs}A_{rs} + \dots.$$

For
$$\begin{aligned} e_s E_s &= e_s \,.\, e_1 \dots e_{s-1} e_{s+1} \dots e_n \\ &= (-1)^{s-1} e_1 \dots e_n = (-1)^{s-1}, \end{aligned}$$

$$e_s E_t = 0 \text{ if } s \text{ is not equal to } t.$$

The factor $(-1)^{r-1}$ on the left is accounted for in the same way.

Thus
$$|a_{nn}| = \sum_s (-)^{r+s} a_{rs} A_{rs}.$$

For example,

$$\begin{vmatrix} a_1, & a_2, & a_3, & a_4 \\ b_1, & b_2, & b_3, & b_4 \\ c_1, & c_2, & c_3, & c_4 \\ d_1, & d_2, & d_3, & d_4 \end{vmatrix} = c_1 \begin{vmatrix} a_2, & a_3, & a_4 \\ b_2, & b_3, & b_4 \\ d_2, & d_3, & d_4 \end{vmatrix} - c_2 \begin{vmatrix} a_1, & a_3, & a_4 \\ b_1, & b_3, & b_4 \\ d_1, & d_3, & d_4 \end{vmatrix}$$

$$+ c_3 \begin{vmatrix} a_1, & a_2, & a_4 \\ b_1, & b_2, & b_4 \\ d_1, & d_2, & d_4 \end{vmatrix} - c_4 \begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ d_1, & d_2, & d_3 \end{vmatrix}.$$

10. In the final equation of Art. 9 A_{rs} is got from $|a_{nn}|$ by erasing the rth row and sth column and writing the remainder as a determinant. It is however more symmetrical, and sometimes convenient, to give to A_{rs} a different form obtained by a series of cyclical permutations of rows and columns.

In A_{rs} remove the first row by a series of interchanges to the last place, then move what is now the first row to the last place, and so on, until we arrive at what was the $(r-1)$th row, which we remove to the last place. This introduces $(r-1)(n-2)$ changes of sign.

Now remove the first column to the last place, and so on, $s-1$ times, necessitating $(s-1)(n-2)$ changes of sign. In all we have introduced

$$(r-1)(n-2)+(s-1)(n-2), \text{ or } (r+s)\,n$$

changes of sign (an even number of changes being neglected). So that, if the new determinant is called A'_{rs}, we have

$$A'_{rs} = (-1)^{n(r+s)} A_{rs},$$

and

$$|a_{nn}| = \Sigma\,(-1)^{(n+1)(r+s)} a_{rs} A'_{rs},$$

where

$$A'_{rs} = \begin{vmatrix} a_{r+1,s+1}, & a_{r+1,s+2} \cdots a_{r+1,n} a_{r+1,1} \cdots a_{r+1,s-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n,s+1}, & a_{n,s+2} \cdots\cdots a_{n,n}, \; a_{n,1} \cdots\cdots a_{n,s-1} \\ a_{1,s+1}, & a_{1,s+2} \cdots\cdots a_{1,n}, \; a_{1,1} \cdots\cdots a_{1,s-1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{r-1,s+1}, & a_{r-1,s+2} \cdots a_{r-1,n} a_{r-1,1} \cdots a_{r-1,s-1} \end{vmatrix}.$$

For example,

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix} = b_1 \begin{vmatrix} c_2, & c_3 \\ a_2, & a_3 \end{vmatrix} + b_2 \begin{vmatrix} c_3, & c_1 \\ a_3, & a_1 \end{vmatrix} + b_3 \begin{vmatrix} c_1, & c_2 \\ a_1, & a_2 \end{vmatrix}.$$

In future we shall always write

$$|a_{nn}| = \sum_s a_{rs} A_{rs},$$

and suppose that A_{rs} has its proper sign.

11. We may arrange the complements of the elements of a determinant in another square array, and then the two arrays

$$\left.\begin{matrix} a_{11} \ldots\ldots a_{1n} \\ \ldots\ldots\ldots\ldots \\ a_{n1} \ldots\ldots a_{nn} \end{matrix}\right\} \ldots\ldots\ldots (1), \qquad \left.\begin{matrix} A_{11} \ldots\ldots A_{1n} \\ \ldots\ldots\ldots\ldots \\ A_{n1} \ldots\ldots A_{nn} \end{matrix}\right\} \ldots\ldots\ldots (2),$$

are said to be reciprocal. (= adjoint)

If now a sum be formed by multiplying each element of a row of (1) by the corresponding element of a row of (2), and adding these products together, the sum is equal to the original determinant or zero, according as the two rows have the same suffix or not. Namely,

$$a_{r1}A_{s1} + a_{r2}A_{s2} + \ldots + a_{rn}A_{sn} = |a_{nn}| \text{ or } 0,$$

according as r is or is not equal to s.

For if r is equal to s the sum on the left is the expansion of the determinant according to the elements of the rth row, but if r is not equal to s the sum on the left is what the expansion of the determinant would be, if its rth and sth rows were identical, but if the elements of two rows are identical the determinant vanishes. In like manner, if we multiply the elements of a column of (1) by the corresponding elements of a column of (2), we get

$$a_{1r}A_{1s} + a_{2r}A_{2s} + \ldots + a_{nr}A_{ns},$$

and this sum is equal to $|a_{nn}|$ or 0, according as r is or is not equal to s.

Kronecker *) has introduced the symbol δ_{rs} to indicate 1 or 0 according as the integers r, s are equal or unequal. With this notation, we have

$$\sum_t a_{tr} A_{ts} = \delta_{rs} |a_{nn}|.$$

*) K. + Hensel, Det. (= B II 1), 1903, p. 228.

12. If all the elements of a row vanish the determinant vanishes, as we see at once by expanding the determinant according to the elements of that row. If all but one vanish the determinant reduces to the product of that element and its complement; viz. if all the elements of the rth row vanish except a_{rs}, then the determinant reduces to $a_{rs}A_{rs}$.

Thus for example,

$$\begin{vmatrix} a_{11}, & a_{12} \dots a_{1n} \\ 0, & a_{22} \dots a_{2n} \\ 0, & a_{32} \dots a_{3n} \\ \dots & \dots \\ 0, & a_{n2} \dots a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} \dots a_{2n} \\ \dots\dots \\ a_{n2} \dots a_{nn} \end{vmatrix}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} \\ 0, & a_{22} & a_{23} \dots a_{2n} \\ 0, & 0, & a_{33} \dots a_{3n} \\ \dots & \dots & \dots \\ 0, & 0, & 0 \dots\dots a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22}, & a_{23} \dots a_{2n} \\ 0, & a_{33} \dots a_{3n} \\ \dots & \dots \\ 0, & 0 \dots\dots a_{nn} \end{vmatrix}$$

$$= a_{11}a_{22} \begin{vmatrix} a_{33} \dots a_{3n} \\ \dots\dots \\ 0 \dots\dots a_{nn} \end{vmatrix}$$

$$= a_{11}a_{22}a_{33} \dots a_{nn}.$$

13. The theorem of the preceding article is of use in evaluating a determinant by reducing it to one of lower order. If the determinant is not of the required form to begin with, it can sometimes be reduced to it. We may exemplify this by finding the value of the determinant

$$D_r = \begin{vmatrix} 0, & a, & a \dots a \\ b, & 0, & a \dots a \\ b, & b, & 0 \dots a \\ \dots & \dots & \dots \\ b, & b, & b \dots 0 \end{vmatrix} (r),$$

the suffixes denoting the order of the determinant. The elements of the leading diagonal are zero, those to the right of it all equal to a, and those to the left all equal to b.

If we subtract each row from the one which follows it, beginning with the last but one,

$$D_r = \begin{vmatrix} 0, & a, & a, & a \ldots\ldots a \\ b, & -a, & 0, & 0 \ldots\ldots 0 \\ 0, & b, & -a, & 0 \ldots\ldots 0 \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \\ 0, & 0, & 0, & 0 \ldots -a \end{vmatrix} (r).$$

The first column contains only one element, hence

$$D_r = -b \begin{vmatrix} a, & a, & a, & a \ldots \\ b, & -a, & 0, & 0 \ldots \\ 0, & b, & -a, & 0 \ldots \\ 0, & 0, & b, & -a \ldots \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{vmatrix} (r-1).$$

Regard the elements in the first row as

$$a+0, \qquad 0+a, \qquad 0+a \ldots$$

then we can resolve the determinant into the sum of two:

$$D_r = -b \begin{vmatrix} a, & 0, & 0, & 0 \ldots \\ b, & -a, & 0, & 0 \ldots \\ 0, & b, & -a, & 0 \ldots \\ 0, & 0, & b, & -a \ldots \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{vmatrix} (r-1) \quad -b \begin{vmatrix} 0, & a, & a, & a \ldots \\ b, & -a, & 0, & 0 \ldots \\ 0, & b, & -a, & 0 \ldots \\ \multicolumn{4}{c}{\cdots\cdots\cdots\cdots\cdots\cdots\cdots} \end{vmatrix} (r-1).$$

In the first of these two determinants all the elements above the leading diagonal vanish, hence its value is $(-1)^{r-2} a^{r-1}$. The second determinant is of the same form as that to which we first reduced D_r, hence

$$D_r = -bD_{r-1} + b(-a)^{r-1}.$$

This is an equation of differences with constant coefficients for D_r, and its solution is

$$D_r = \frac{(-1)^{r-1} ab}{a-b} (a^{r-1} - b^{r-1}).$$

14. In Art. 11 we saw how under certain circumstances the order

of a determinant might be reduced. Conversely we are enabled to increase the order of a determinant without altering its value, namely, by bordering it with a new row and column in one of which all the elements vanish except that common to the other. Thus

$$|a_{nn}| = \begin{vmatrix} 1, & 0, & 0, & 0 & \ldots \\ x, & a_{11}, & a_{12}, & a_{13} & \ldots \\ y, & a_{21}, & a_{22}, & a_{23} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} 0, & 0, & 0 & \ldots & 0, & 1 \\ a_{11}, & a_{12}, & a_{13} & \ldots & a_{1n}, & x \\ a_{21}, & a_{22}, & a_{23} & \ldots & a_{2n}, & y \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

where the quantities x, y ... are any whatever. By adding on to these a new row and column we can raise the order of the determinant to $n + 2$ and so on.

15. In the determinant $D = |a_{nn}|$, if we suppose only the element a_{rs} to vary, since on expanding according to the elements of the rth row

$$D = a_{r1}A_{r1} + a_{r2}A_{r2} + \ldots + a_{rs}A_{rs} + \ldots$$

the only variable term on the right is the product $a_{rs}A_{rs}$, we see at once that

$$\frac{\partial D}{\partial a_{rs}} = A_{rs}.$$

If among the elements of A_{rs} only a_{fg} is variable, we see that

$$\frac{\partial A_{rs}}{\partial a_{fg}} = \frac{\partial^2 D}{\partial a_{fg} \partial a_{rs}}.$$

Thus $\dfrac{\partial^2 D}{\partial a_{fg} \partial a_{rs}} a_{fg} a_{rs}$ is the sum of all terms in D which contain the product $a_{fg} a_{rs}$.

The differential coefficient

$$\frac{\partial^2 D}{\partial a_{fg} \partial a_{rs}}$$

is the determinant obtained by erasing in D the rth and fth rows and the sth and gth columns, and is therefore complementary to

$$\begin{vmatrix} a_{rs}, & a_{rg} \\ a_{fs}, & a_{fg} \end{vmatrix}.$$

In like manner it is plain that

$$\frac{\partial^{n-m} D}{\partial a_{fr} \partial a_{gs} \dots} \text{ and } \frac{\partial^m D}{\partial a_{pu} \partial a_{qv} \dots}$$

are complementary determinants if

$$f,\ g \dots p,\ q \dots$$
$$r,\ s \dots u,\ v \dots$$

are each of them permutations of $1, 2 \dots n$, i.e. if the product

$$a_{fr} a_{gs} \dots a_{pu} a_{qv} \dots$$

is a term of the determinant D.

16. If all the elements of a determinant are functions of a variable t we see that

$$\frac{dD}{dt} = \Sigma \frac{\partial D}{\partial a_{rs}} \cdot \frac{da_{rs}}{dt} \quad (r, s = 1, 2 \dots n).$$

If we denote differential coefficients with respect to t by accents we have

$$D' = \Sigma A_{r1} a'_{r1} + \Sigma A_{r2} a'_{r2} + \dots$$

$$= \begin{vmatrix} a'_{11}, & a_{12} \dots a_{1n} \\ a'_{21}, & a_{22} \dots a_{2n} \\ \dots\dots & \dots\dots \end{vmatrix} + \begin{vmatrix} a_{11}, & a'_{12}, \dots a_{1n} \\ a_{21}, & a'_{22}, \dots a_{2n} \\ \dots\dots & \dots\dots \end{vmatrix} + \dots$$

So that D' is the sum of n determinants obtained by substituting for the elements of each column of D in succession their differential coefficients with respect to t.

An interesting example of this is to consider the differential coefficient of

$$D = \begin{vmatrix} u, & u', & u'', & \dots & u^{(n-1)} \\ v, & v', & v'', & \dots & v^{(n-1)} \\ w, & w', & w'', & \dots & w^{(n-1)} \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix},$$

accents denoting differential coefficients with respect to t.

Each of the first $n-1$ determinants obtained by the preceding rule vanishes because it has two columns alike, the last alone does not vanish, so that

$$\frac{dD}{dt} = \begin{vmatrix} u, & u' \dots u^{(n-2)}, & u^{(n)} \\ v, & v' \dots v^{(n-2)}, & v^{(n)} \\ w, & w' \dots w^{(n-2)}, & w^{(n)} \\ \dots & \dots & \dots \end{vmatrix}.$$

As another example take the determinant

$$D_n = \begin{vmatrix} 1, & 1 & \dots & 1 \\ t_1, & t_2 & \dots & t_n \\ t_1^2, & t_2^2 & \dots & t_n^2 \\ \dots & \dots & \dots & \dots \\ t_1^{n-1}, & t_2^{n-1} & \dots & t_n^{n-1} \end{vmatrix}.$$

Then $\dfrac{\partial D_n}{\partial t_r}$ is got from D_n by substituting for the elements of the rth column

$$0,\ 1,\ 2t_r,\ 3t_r^2 \dots (n-1)\, t_r^{n-2}.$$

Hence

$$\frac{\partial^{n-1} D_n}{\partial t_1 \partial t_2 \dots \partial t_{n-1}} = \begin{vmatrix} 0, & 0 & \dots & 0, & 1 \\ 1, & 1 & \dots & 1, & t_n \\ 2t_1, & 2t_2 & \dots & 2t_{n-1}, & t_n^2 \\ \dots & \dots & \dots & \dots & \dots \\ (n-1)\, t_1^{n-2}, & (n-1)\, t_2^{n-2} & \dots & (n-1)\, t_{n-1}^{n-2}, & t_n^{n-1} \end{vmatrix}$$

$$= (-1)^{n-1} (n-1)!\, D_{n-1}.$$

17. We may use the theorems of Art. 11 of the present chapter to prove those of Arts. 3 and 4 of Chap. III.

If each element of a row of a determinant is the sum of p terms, the determinant is equal to the sum of p determinants having for their elements the separate terms of the sum in question.

For if $\qquad a_{rs} = p_s + q_s + \dots + t_s,$

then
$$\begin{aligned} |a_{nn}| &= \Sigma a_{rs} A_{rs} \\ &= \Sigma p_s A_{rs} + \Sigma q_s A_{rs} + \dots + \Sigma t_s A_{rs} \\ &= P + Q + \dots + T, \end{aligned}$$

where P is the determinant obtained from the given one by writing $p_1, p_2 \ldots p_n$ for the elements of the rth row and $Q \ldots T$ have similar meanings.

The value of a determinant is not altered by adding to the elements of any row those of another row multiplied by a constant factor. For if to the elements of the rth row we add those of the tth row, each multiplied by p, the resulting determinant is equal to

$$\begin{aligned} \sum_s (a_{rs} + pa_{ts}) A_{rs} &= \Sigma a_{rs} A_{rs} + p \Sigma a_{ts} A_{rs} \\ &= |a_{nn}|, \end{aligned}$$

the last sum vanishing by Art. 11.

18. If each element of a determinant consists of the sum of p terms, we could by continued application of the first theorem in Art. 17 reduce this determinant to a sum of determinants whose elements are all single terms. But a formula of expansion has been given by Albeggiani which presents the result in a more suitable form for applications.

Let $$a_{rs} = a_{rs1} + a_{rs2} + \ldots + a_{rsp},$$

so that each element in the determinant is the sum of p terms. Then each column of the determinant when written at full length would consist of p partial columns whose suffixes are the third suffixes of the above elements. With these partial columns we can form p determinants, taking all the partial columns with the third suffix 1 to form the first, those with the third suffix 2 to form the second, and so on. We shall denote these determinants by

$$D_1^{(n)},\ D_2^{(n)} \ldots D_p^{(n)},$$

so that

$$D_u^{(n)} = \begin{vmatrix} a_{11u}, & a_{12u} & \ldots & a_{1nu} \\ a_{21u}, & a_{22u} & \ldots & a_{2nu} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1u}, & a_{n2u} & \ldots & a_{nnu} \end{vmatrix}.$$

The first two suffixes tell us the row and column in which the element stands, the third the determinant to which it belongs. The original determinant is denoted by $D^{(n)}$. The index in brackets tells us the order of the determinant.

19. We shall find it necessary to employ the term complementary minors in the following sense. From the elements of $D_1^{(n)}$, form a minor $D_1^{(\alpha)}$ of order α by selecting α rows and columns. Then in $D_2^{(n)}$ select β rows and columns, whose suffixes are different from those selected to form $D_1^{(\alpha)}$, these form a determinant $D_2^{(\beta)}$, and so on until we take π rows and columns from $D_p^{(n)}$, to form a determinant $D_p^{(\pi)}$, none of which have the same suffix as any of the preceding. Then if

$$\alpha + \beta + \gamma + \ldots + \pi = n \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

$$D_1^{(\alpha)},\ D_2^{(\beta)},\ D_3^{(\gamma)} \ldots D_p^{(\pi)}$$

shall be called a series of complementary minors. Any one or more of the numbers $\alpha, \beta \ldots \pi$ may be unity or zero.

20. We shall now prove that

$$D^{(n)} = S\Sigma D_1^{(\alpha)} D_2^{(\beta)} \ldots D_p^{(\pi)},$$

where the meanings of the summation signs will be explained presently. For we have

$$D^{(n)} = \Pi\,(a_{r1}e_1 + a_{r2}e_2 + \ldots + a_{rn}e_n),$$

and if
$$u_{rs} = a_{r1s}e_1 + a_{r2s}e_2 + \ldots + a_{rns}e_n,$$

$$D^{(n)} = \Pi\,(u_{r1} + u_{r2} + \ldots + u_{rp}) \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

the product containing n factors.

We shall obtain a term of the product on the right if we take α factors such as u_{r1}, β factors such as $u_{r2}, \ldots$ π factors such as u_{rp}, provided the equation (1) is satisfied.

But from the definition of a determinant this product of factors is equal to a determinant of order n the first α of whose rows come from $D_1^{(n)}$, the next β from $D_2^{(n)}, \ldots$ the last π from $D_p^{(n)}$. Expand this determinant in the sum of products of complementary minors of order $\alpha, \beta \ldots \pi$ selecting the rows of the minors from the first α, the next $\beta, \ldots$ the last π, its value is then (Art. 8)

$$\Sigma D_1^{(\alpha)} D_2^{(\beta)} \ldots D_p^{(\pi)},$$

with the notation of Art. 19, and the summation sign means that we are to take all the possible complementary minors.

This is only a single term in the expansion of the product (2); the whole product is obtained by summing this for all values of $\alpha, \beta \ldots \pi$ which satisfy the equation (1).

Thus $$D^{(n)} = S\Sigma D_1^{(\alpha)} D_2^{(\beta)} \ldots D_p^{(\pi)} \quad \text{.............. (3).}$$

21. The number of terms in the sum Σ is

$$\frac{n!}{\alpha! \beta! \ldots \pi!}.$$

Let us compare the expansion (3) with the expansion of the multinomial

$$(D_1 + D_2 + \ldots + D_p)^n.$$

The general term is

$$CD_1^{\alpha} D_2^{\beta} \ldots D_p^{\pi} \quad \text{...................... (4),}$$

where $\alpha, \beta \ldots \pi$ satisfy (1) and

$$C = \frac{n!}{\alpha! \beta! \ldots \pi!}.$$

Comparing (3) and (4) we see that in expanding the determinant we replace C by Σ, and $\alpha, \beta \ldots \pi$ are no longer exponents, but merely indicate the orders of the determinants $D_1^{(\alpha)}$, $D_2^{(\beta)}$, etc.

Hence we may write symbolically for the expansion of our determinant

$$(D_1 + D_2 + \ldots + D_p)^n,$$

where in every term of the multinomial expansion we replace the coefficient by a summation sign, the number of terms in the sum being given by the multinomial coefficient and the exponents $\alpha, \beta \ldots \pi$ now indicating the orders of the complementary minors. Thus finally we have the symbolical equation

$$D^{(n)} = (D_1 + D_2 + \ldots + D_p)^n.$$

22. Let us make use of this theorem to expand the determinant

$$D = \begin{vmatrix} a_{11} + z_1, & a_{12}\;, & a_{13} & \ldots & a_{1n} \\ a_{21}\;, & a_{22} + z_2, & a_{23} & \ldots & a_{2n} \\ a_{31}\;, & a_{32}\;, & a_{33} + z_3 & \ldots & a_{3n} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1}\;, & a_{n2}\;, & a_{n3} & \ldots & a_{nn} + z_n \end{vmatrix}$$

according to products of the quantities $z_1, z_2 \ldots z_n$.

Here we must write

$$D_1^{(n)} = \begin{vmatrix} a_{11} \dots a_{1n} \\ \dots\dots\dots\dots \\ a_{n1} \dots a_{nn} \end{vmatrix}, \qquad D_2^{(n)} = \begin{vmatrix} z_1, & 0 \dots 0 \\ 0, & z_2 \dots 0 \\ \dots & \dots\dots \\ 0, & 0 \dots z_n \end{vmatrix}.$$

Then by the above theorem

$$D = (D_1 + D_2)^n$$
$$= D_1^{(n)} + \Sigma D_1^{(n-1)} D_2^{(1)} + \Sigma D_1^{(n-2)} D_2^{(2)} + \dots + D_2^{(n)}.$$

Now clearly all minors of $D_2^{(n)}$ vanish except those whose leading diagonal is part of the leading diagonal of $D_2^{(n)}$.

Thus

$$D_2^{(1)} = z_i, \; D_2^{(2)} = z_i z_k, \; \dots \; D_2^{(n)} = z_1 z_2 \dots z_n.$$

The corresponding minors $D_1^{(n-1)}, D_1^{(n-2)} \dots$ are got by erasing in $D_1^{(n)}$ the ith row and column, the ith and kth rows and columns, &c.

Thus

$$D = D_1^{(n)} + \Sigma z_i D_1^{(n-1)} + \Sigma z_i z_k D_1^{(n-2)} + \dots + z_1 z_2 \dots z_n.$$

Or if we simply denote $D_1^{(n)}$ by D_1,

$$D = D_1 + \Sigma z_i \frac{\partial D_1}{\partial a_{ii}} + \Sigma z_i z_k \frac{\partial^2 D_1}{\partial a_{ii} \partial a_{kk}} + \dots + z_1 z_2 \dots z_n.$$

If $z_1 = z_2 \dots = z_n$ we get

$$D = D_1 + z \Sigma \frac{\partial D_1}{\partial a_{ii}} + z^2 \Sigma \frac{\partial^2 D_1}{\partial a_{ii} \partial a_{kk}} + \dots + z^n.$$

These results may also be obtained by using the generalised form of Taylor's theorem.

23. Any determinant can be written in the form

$$D = \begin{vmatrix} 0 + a_{11}, & a_{12} & \dots & a_{1n} \\ a_{21}, & 0 + a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2} & \dots & 0 + a_{nn} \end{vmatrix}.$$

We may now apply the theorem of Art. 22 by supposing

$$D_1 = \begin{vmatrix} 0, & a_{12} \dots a_{1n} \\ a_{21}, & 0 \dots a_{2n} \\ \dots & \dots\dots \\ a_{n1}, & a_{n2} \dots 0 \end{vmatrix}$$

and

$$z_i = a_{ii}.$$

Then

$$D = D_1 + \Sigma a_{ii}\frac{\partial D_1}{\partial a_{ii}} + \Sigma a_{ii}a_{kk}\frac{\partial^2 D_1}{\partial a_{ii}\partial a_{kk}} + \dots + a_{11}a_{22}\dots a_{nn},$$

the general term being

$$\Sigma a_{ii}a_{kk}\dots a_{rr}D_1^{(n-m)},$$

where $D_1^{(n-m)}$ is the minor obtained from D_1 by suppressing the ith, kth ... rth rows and columns, m in number.

It is clear that $D_1^{(1)}$ is zero, for the suppression of $(n-1)$ corresponding rows and columns of D_1 leaves us with one of the zero elements in the leading diagonal.

Ex. If

$$\begin{vmatrix} 0, & a_{12} \\ a_{21}, & 0 \end{vmatrix} = (12), \text{\&c.}$$

we have

$$\begin{vmatrix} a_{11} & \dots & a_{14} \\ \dots & \dots & \dots \\ a_{41} & \dots & a_{44} \end{vmatrix} = \begin{aligned}[t] & a_{11}a_{22}a_{33}a_{44} + a_{11}a_{22}(34) + a_{11}a_{33}(24) + a_{11}a_{44}(23) \\ & + a_{22}a_{33}(14) + a_{22}a_{44}(13) + a_{33}a_{44}(12) \\ & + a_{11}(234) + a_{22}(134) + a_{33}(124) + a_{44}(123) \\ & + (1234). \end{aligned}$$

As another example we may find the value of the determinant

$$D = \begin{vmatrix} c_1, & a, & a, & a \dots a \\ b, & c_2, & a, & a \dots a \\ b, & b, & c_3, & a \dots a \\ \dots & \dots & \dots & \dots \\ b, & b, & b, & b \dots c_n \end{vmatrix}.$$

The general term in the expansion of this determinant is

$$\Sigma c_i c_k \dots c_r D_1^{(n-m)},$$

where $c_i, c_k \dots c_r$ are any m elements of the leading diagonal. But by Art. 13

$$D_1^{(n-m)} = (-1)^{n-m-1}\frac{ab}{a-b}(a^{n-m-1} - b^{n-m-1});$$

whence, if $\quad f(x) = (c_1 - x)(c_2 - x)\dots(c_n - x),$

it is clear that

$$D = \frac{af(b) - bf(a)}{a-b}.$$

If we write down the similar determinant of order $n+1$, for which $c_{n+1}=0$, after dividing both sides by ab, we get

$$\begin{vmatrix} c_1, & a & \dots & a, & 1 \\ b, & c_2 & \dots & a, & 1 \\ \dots & & & & \\ b, & b & \dots & c_n, & 1 \\ 1, & 1 & \dots & 1, & 0 \end{vmatrix} = \frac{f(a)-f(b)}{a-b}.$$

If we now put $b=a$, we get a determinant expression for $f'(a)$.

24. We have seen how to expand a determinant according to the elements of a row or column. It is frequently useful to be able to expand a determinant according to the elements of a row and column. This is effected by means of the following theorem due to Cauchy,

$$|a_{nn}| = a_{rs}A_{rs} - \Sigma a_{rk}a_{is}B_{ik},$$

which expands a determinant as an explicit function of the elements which occupy the rth row and sth column.

A_{rs} is the complement of a_{rs} and B_{ik} is the complement of a_{ik} in A_{rs}, and is therefore a second minor of the original determinant.

For every term which does not contain a_{rs} must contain some other element from the rth row and some other element from the sth column, and hence contains such a product as $a_{rk}a_{is}$, where i and k are different from r and s respectively. The aggregate of all terms which multiply a_{rs} is A_{rs}; now $a_{rk}a_{is}$ differs from $a_{rs}a_{ik}$ by the interchange of the suffixes k and s, thus the aggregate of terms which multiplies $a_{rk}a_{is}$ differs in sign only from that which multiplies $a_{rs}a_{ik}$, that is to say, differs in sign only from the coefficient of a_{ik} in A_{rs}. Hence $-B_{ik}$ is the coefficient in question.

25. This theorem is useful for expanding a determinant which has been bordered. For example by this theorem

$$D = \begin{vmatrix} b_{pq}, & b_{p1}, & b_{p2} \dots \\ b_{1q}, & a_{11}, & a_{12} \dots \\ b_{2q}, & a_{21}, & a_{22} \dots \\ \dots & & \end{vmatrix}$$
$$= b_{pq}\,|\,a_{nn}\,| - \Sigma b_{pk}b_{iq}A_{ik},$$

where A_{ik} is the complement of a_{ik} in $|\,a_{nn}\,|$.

By the selection of a suitable bordering we are often able to evaluate a determinant by means of this theorem.

For example, let

$$D = \begin{vmatrix} x_1, & a_2, & a_3 & \dots & a_n \\ a_1, & x_2, & a_3 & \dots & a_n \\ a_1, & a_2, & x_3 & \dots & a_n \\ \dots & \dots & \dots & \dots & \dots \\ a_1, & a_2, & a_3 & \dots & x_n \end{vmatrix},$$

all the elements in the ith column being a_i except that in the ith row, which is x_i.

Then by Art. 14

$$D = \begin{vmatrix} 1, & 0, & 0, & 0 & \dots \\ 1, & x_1, & a_2, & a_3 & \dots \\ 1, & a_1, & x_2, & a_3 & \dots \\ 1, & a_1, & a_2, & x_3 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Multiply the first column by a_i, and subtract it from the ith column; do this for each column, the value of the determinant is unaltered, and

$$D = \begin{vmatrix} 1, & -a_1, & -a_2, & -a_3, & \dots \\ 1, & x_1 - a_1, & 0, & 0, & \dots \\ 1, & 0, & x_2 - a_2, & 0, & \dots \\ 1, & 0, & 0, & x_3 - a_3, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

Here the bordered determinant is

$$\left| \begin{matrix} x_1 - a_1, & 0, & 0 & \dots \\ 0, & x_2 - a_2, & 0 & \dots \\ 0, & 0, & x_3 - a_3 & \dots \\ \dots & \dots & \dots & \dots \end{matrix} \right|$$

for which all first minors vanish except those of diagonal elements.

Hence, in the theorem of this article, we must suppose $i = k$; if

$$f = (x_1 - a_1)(x_2 - a_2) \dots (x_n - a_n),$$

$$f'(x_r) = \frac{\partial f}{\partial x_r},$$

it follows that

$$D = f + \Sigma a_r f'(x_r),$$

a theorem due to Sardi.

CHAPTER V.

COMPOSITION OF ARRAYS. MULTIPLICATION OF DETERMINANTS.

1. In dealing with rectangular arrays it is often convenient to use an abbreviated notation. The array

$$\begin{array}{l} a_{11}, \ a_{12} \ \dots \ a_{1n} \\ a_{21}, \ a_{22} \ \dots \ a_{2n} \\ \dots\dots\dots\dots\dots\dots \\ a_{m1}, \ a_{m2} \dots \ a_{mn} \end{array}$$

with m rows and n columns is said to be of the type $m \times n$, and may be denoted by the symbol (a_{mn}).

Associated with this is the array

$$\begin{array}{l} a_{11}, \ a_{21} \ \dots \ a_{m1} \\ a_{12}, \ a_{22} \ \dots \ a_{m2} \\ \dots\dots\dots\dots\dots\dots \\ a_{1n}, \ a_{2n} \ \dots \ a_{mn} \end{array}$$

which is called the conjugate of (a_{mn}), and will be denoted by $(a_{mn})'$. This is of type $n \times m$, and its conjugate is (a_{mn}).

A square array of type $n \times n$ in which all the elements are zero except those in the leading diagonal may be denoted by $[a_{nn}]$. Another way of writing it is $(\delta_{nn} a_{nn})$.

The array (a_{mn}) is said to be deficient if $m < n$; redundant if $m > n$. When $m = n$ we have a square array. The conjugate of a redundant array is deficient, and *vice versa*.

Two arrays of the same type may be combined into a sum or difference according to the rules expressed by

$$(a_{mn}) \pm (b_{mn}) = (a_{mn} \pm b_{mn}).$$

If k is any constant quantity, the product of k and (a_{mn}) is defined by the formula

$$k \times (a_{mn}) = (a_{mn}) \times k = (ka_{mn}).$$

2. Suppose, now, that (a_{mn}), (b_{np}) are any two rectangular arrays of types $m \times n$ and $n \times p$ respectively. Let

$$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk};$$

then there are $m \times p$ quantities c_{ik}, which may be regarded as the elements of an array (c_{mp}). We shall write

$$(c_{mp}) = (a_{mn})\ (b_{np}).$$

It is to be carefully observed that the order of factors on the right is essential. According to the definition $(b_{np})\ (a_{mn})$ has no meaning unless $m = p$; and even then the meaning of the symbol is, in general, different from that of $(a_{mn})\ (b_{np})$.

3. When $p = m$, the array (c_{mp}) is square; we propose to find an expression for the determinant $|c_{mm}|$.

Taking m alternate units $e_1, e_2, \ldots e_m$, we have

$$C_i = c_{i1}e_1 + c_{i2}e_2 + \ldots + c_{im}e_m = a_{i1}B_1 + a_{i2}B_2 + \ldots + a_{in}B_n,$$

where $$B_k = b_{k1}e_1 + b_{k2}e_2 + \ldots + b_{km}e_m.$$

Hence

$$|c_{mm}| = \Pi C_i = \Pi\ (a_{i1}B_1 + a_{i2}B_2 + \ldots + a_{in}B_n).$$

There are now three cases to consider:

(i) If $m > n$, the product last written vanishes, because in each term of the expansion at least one factor B_k is repeated.

(ii) If $m = n$,

$$|c_{nn}| = |a_{nn}|\ \Pi B_k = |a_{nn}| \cdot |b_{nn}|.$$

(iii) If $m < n$, the product on the right is the sum of such terms as

$$\begin{vmatrix} a_{1p}, & a_{1q}, & a_{1r} & \ldots \\ a_{2p}, & a_{2q}, & a_{2r} & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ a_{mp}, & a_{mq}, & a_{mr} & \ldots \end{vmatrix} B_pB_qB_r\ldots,$$

where $p, q, r \ldots$ are m numbers taken from $1, 2 \ldots n$ (IV. 5).

But

$$B_pB_qB_r \ldots = \begin{vmatrix} b_{p1}, & b_{p2} \ldots b_{pm} \\ b_{q1}, & b_{q2} \ldots b_{qm} \\ b_{r1}, & b_{r2} \ldots b_{rm} \\ \multicolumn{2}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \end{vmatrix} e_1e_2 \ldots e_m,$$

so that, finally,

$$|c_{mm}| = \Sigma \begin{vmatrix} a_{1p}, & a_{1q}, & a_{1r} & \ldots \\ a_{2p}, & a_{2q}, & a_{2r} & \ldots \\ \multicolumn{4}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ a_{mp}, & a_{mq}, & a_{mr} & \ldots \end{vmatrix} \cdot \begin{vmatrix} b_{p1}, & b_{p2} \ldots b_{pm} \\ b_{q1}, & b_{q2} \ldots b_{qm} \\ b_{r1}, & b_{r2} \ldots b_{rm} \\ \multicolumn{2}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \end{vmatrix}$$

where for $p, q, r \ldots$ we are to write all possible m-ads from the n numbers $1, 2 \ldots n$.

4. The second case of Art. 3 gives us a rule for expressing the product of two determinants of the nth order as another determinant of the same order; the rule being

$$|a_{nn}| \cdot |b_{nn}| = |c_{nn}|,$$

where $$c_{ik} = a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}.$$

Since the element c_{ik} is derived from the ith row of $|a_{nn}|$ and the kth column of $|b_{nn}|$ the product formed by this rule is said to be effected according to the rows of the first determinant and columns of the second.

But since in either or both of the determinants $|a_{nn}|$, $|b_{nn}|$ we may interchange rows and columns without affecting their value, we see that the product of two determinants can be obtained in the form of a determinant in four different ways, viz. the element c_{ik} has one of the four forms:

$$\begin{gathered} a_{i1}b_{1k} + a_{i2}b_{2k} + \ldots + a_{in}b_{nk}, \\ a_{i1}b_{k1} + a_{i2}b_{k2} + \ldots + a_{in}b_{kn}, \\ a_{1i}b_{1k} + a_{2i}b_{2k} + \ldots + a_{ni}b_{nk}, \\ a_{1i}b_{k1} + a_{2i}b_{k2} + \ldots + a_{ni}b_{kn}, \end{gathered}$$

where we multiply the elements of a row of $|a_{nn}|$ by the corresponding elements of a row or column of $|b_{nn}|$; or the elements of a column of $|a_{nn}|$ by the corresponding elements of a column or row of $|b_{nn}|$. There are really only two essentially distinct cases: multiplying by rows, when we multiply corresponding elements of

two rows together; and multiplying by rows and columns, when we multiply the elements of a row by the corresponding elements of a column.

The four forms of the product correspond to the four compositions $(a_{nn})(b_{nn})$, $(a_{nn})(b_{nn})'$, $(a_{nn})'(b_{nn})$, $(a_{nn})'(b_{nn})'$. The product of two determinants of orders n and m $(n > m)$ can be expressed as a determinant of order n by applying the process of IV. 14 to increase the order of one of them until it is equal to that of the other.

5. *Examples.* Compounding the two systems

$$\begin{matrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{matrix} \qquad \begin{matrix} p_1, & p_2 \\ q_1, & q_2 \\ r_1, & r_2 \end{matrix}$$

we get the theorem

$$\begin{vmatrix} a_1p_1 + b_1q_1 + c_1r_1, & a_1p_2 + b_1q_2 + c_1r_2 \\ a_2p_1 + b_2q_1 + c_2r_1, & a_2p_2 + b_2q_2 + c_2r_2 \end{vmatrix}$$

$$= \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix} \cdot \begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix} + \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix} \cdot \begin{vmatrix} p_1, & r_1 \\ p_2, & r_2 \end{vmatrix} + \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix} \cdot \begin{vmatrix} q_1, & r_1 \\ q_2, & r_2 \end{vmatrix},$$

while if we compound the systems

$$\begin{matrix} a_1, & a_2 \\ b_1, & b_2 \\ c_1, & c_2 \end{matrix} \qquad \begin{matrix} p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \end{matrix}$$

we get

$$\begin{vmatrix} a_1p_1 + a_2p_2, & a_1q_1 + a_2q_2, & a_1r_1 + a_2r_2 \\ b_1p_1 + b_2p_2, & b_1q_1 + b_2q_2, & b_1r_1 + b_2r_2 \\ c_1p_1 + c_2p_2, & c_1q_1 + c_2q_2, & c_1r_1 + c_2r_2 \end{vmatrix} = 0.$$

Again, the product of the two determinants

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} \qquad \begin{vmatrix} p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \\ p_3, & q_3, & r_3 \end{vmatrix}$$

is the determinant

$$\begin{vmatrix} a_1p_1+b_1q_1+c_1r_1, & a_1p_2+b_1q_2+c_1r_2, & a_1p_3+b_1q_3+c_1r_3 \\ a_2p_1+b_2q_1+c_2r_1, & a_2p_2+b_2q_2+c_2r_2, & a_2p_3+b_2q_3+c_2r_3 \\ a_3p_1+b_3q_1+c_3r_1, & a_3p_2+b_3q_2+c_3r_2, & a_3p_3+b_3q_3+c_3r_3 \end{vmatrix};$$

while

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} \cdot \begin{vmatrix} p_1, & q_1 \\ p_2, & q_2 \end{vmatrix} = \begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} \begin{vmatrix} p_1, & q_1, & 0, & 0 \\ p_2, & q_2, & 0, & 0 \\ 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix}$$

(forming the product by rows and columns)

$$= \begin{vmatrix} a_1p_1+b_1p_2, & a_1q_1+b_1q_2, & c_1, & d_1 \\ a_2p_1+b_2p_2, & a_2q_1+b_2q_2, & c_2, & d_2 \\ a_3p_1+b_3p_2, & a_3q_1+b_3q_2, & c_3, & d_3 \\ a_4p_1+b_4p_2, & a_4q_1+b_4q_2, & c_4, & d_4 \end{vmatrix}.$$

Multiplying by rows we have

$$\begin{vmatrix} a, & b \\ -b', & a' \end{vmatrix} \begin{vmatrix} c, & d \\ -d', & c' \end{vmatrix} = \begin{vmatrix} ac+bd, & -ad'+bc' \\ -b'c+a'd, & b'd'+a'c' \end{vmatrix}.$$

Now let a, b, c, d be the complex numbers

$$\begin{array}{lll} a = x+iy & b = u+iv & \\ & & i=\sqrt{-1}, \\ c = p+iq & d = r+is & \end{array}$$

and a', b', c', d' their conjugates, $a' = x - iy$, &c. On multiplying out the three determinants we have Euler's theorem concerning the product of two numbers each the sum of four squares, viz.

$$(x^2+y^2+u^2+v^2)(p^2+q^2+r^2+s^2),$$
$$= (px-qy+ru-sv)^2 + (py+qx+rv+su)^2$$
$$+ (pu+qv-rx-sy)^2 + (pv-qu-ry+sx)^2.$$

6. The square array $(a_{mn})(a_{mn})'$ has for its elements

$$c_{ik} = a_{i1}a_{k1} + a_{i2}a_{k2} + \ldots + a_{in}a_{kn} = c_{ki},$$

and, if $m \leqslant n$,

$$|c_{mm}| = \Sigma \begin{vmatrix} a_{1p}, & a_{1q}, & a_{1r} & \dots \\ a_{2p}, & a_{2q}, & a_{2r} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}^2,$$

or the determinant is the sum of n_m squares. If then the elements a_{ik} are all real the determinant $|c_{mm}|$ can only vanish when the determinant under the summation sign on the right vanishes for all values of $p, q, r \dots$.

Thus compounding

$$\begin{matrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \end{matrix}$$

with its conjugate, we see that

$$\begin{vmatrix} a_1^2 + b_1^2 + c_1^2, & a_1a_2 + b_1b_2 + c_1c_2 \\ a_1a_2 + b_1b_2 + c_1c_2, & a_2^2 + b_2^2 + c_2^2 \end{vmatrix} = \begin{vmatrix} a_1, & b_1 \\ a_2, & b_2 \end{vmatrix}^2 + \begin{vmatrix} b_1, & c_1 \\ b_2, & c_2 \end{vmatrix}^2 + \begin{vmatrix} a_1, & c_1 \\ a_2, & c_2 \end{vmatrix}^2,$$

or

$$(a_1^2 + b_1^2 + c_1^2)(a_2^2 + b_2^2 + c_2^2) - (a_1a_2 + b_1b_2 + c_1c_2)^2$$
$$= (a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (a_1c_2 - a_2c_1)^2.$$

Again

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}^2 = \begin{vmatrix} a_1^2 + b_1^2 + c_1^2, & a_1a_2 + b_1b_2 + c_1c_2, & a_1a_3 + b_1b_3 + c_1c_3 \\ a_1a_2 + b_1b_2 + c_1c_2, & a_2^2 + b_2^2 + c_2^2, & a_2a_3 + b_2b_3 + c_2c_3 \\ a_1a_3 + b_1b_3 + c_1c_3, & a_2a_3 + b_2b_3 + c_2c_3, & a_3^2 + b_3^2 + c_3^2 \end{vmatrix}.$$

7. Sylvester has shewn how, by the artifice of bordering the determinants as in IV. 14, the product of two determinants of order n can be represented in $n+1$ distinct forms. We shall illustrate this for the case $n = 3$.

The product of the two determinants

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}, \qquad \begin{vmatrix} p_1, & q_1, & r_1 \\ p_2, & q_2, & r_2 \\ p_3, & q_3, & r_3 \end{vmatrix},$$

is the determinant of order 3:

$$\begin{vmatrix} a_1p_1 + b_1q_1 + c_1r_1, & a_1p_2 + b_1q_2 + c_1r_2, & a_1p_3 + b_1q_3 + c_1r_3 \\ a_2p_1 + b_2q_1 + c_2r_1, & a_2p_2 + b_2q_2 + c_2r_2, & a_2p_3 + b_2q_3 + c_2r_3 \\ a_3p_1 + b_3q_1 + c_3r_1, & a_3p_2 + b_3q_2 + c_3r_2, & a_3p_3 + b_3q_3 + c_3r_3 \end{vmatrix}.$$

But if before forming their product we write the determinants in the respective forms

$$\begin{vmatrix} a_1, & b_1, & c_1, & 0 \\ a_2, & b_2, & c_2, & 0 \\ a_3, & b_3, & c_3, & 0 \\ 0, & 0, & 0, & 1 \end{vmatrix} \qquad - \begin{vmatrix} p_1, & q_1, & 0, & r_1 \\ p_2, & q_2, & 0, & r_2 \\ p_3, & q_3, & 0, & r_3 \\ 0, & 0, & 1, & 0 \end{vmatrix},$$

their product by rows is the determinant of order 4:

$$- \begin{vmatrix} a_1p_1 + b_1q_1, & a_1p_2 + b_1q_2, & a_1p_3 + b_1q_3, & c_1 \\ a_2p_1 + b_2q_1, & a_2p_2 + b_2q_2, & a_2p_3 + b_2q_3, & c_2 \\ a_3p_1 + b_3q_1, & a_3p_2 + b_3q_2, & a_3p_3 + b_3q_3, & c_3 \\ r_1, & r_2, & r_3, & 0 \end{vmatrix}.$$

Again writing the original determinants in the forms

$$\begin{vmatrix} a_1, & b_1, & c_1, & 0, & 0 \\ a_2, & b_2, & c_2, & 0, & 0 \\ a_3, & b_3, & c_3, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix}, \qquad \begin{vmatrix} p_1, & 0, & 0, & q_1, & r_1 \\ p_2, & 0, & 0, & q_2, & r_2 \\ p_3, & 0, & 0, & q_3, & r_3 \\ 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0 \end{vmatrix},$$

their product is now the determinant of order 5:

$$\begin{vmatrix} a_1p_1, & a_1p_2, & a_1p_3, & b_1, & c_1 \\ a_2p_1, & a_2p_2, & a_2p_3, & b_2, & c_2 \\ a_3p_1, & a_3p_2, & a_3p_3, & b_3, & c_3 \\ q_1, & q_2, & q_3, & 0, & 0 \\ r_1, & r_2, & r_3, & 0, & 0 \end{vmatrix};$$

while writing the determinants in the forms

$$\begin{vmatrix} a_1, & b_1, & c_1, & 0, & 0, & 0 \\ a_2, & b_2, & c_2, & 0, & 0, & 0 \\ a_3, & b_3, & c_3, & 0, & 0, & 0 \\ 0, & 0, & 0, & 1, & 0, & 0 \\ 0, & 0, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & 0, & 0, & 1 \end{vmatrix}, \qquad \begin{vmatrix} 1, & 0, & 0, & 0, & 0, & 0 \\ 0, & 1, & 0, & 0, & 0, & 0 \\ 0, & 0, & 1, & 0, & 0, & 0 \\ 0, & 0, & 0, & p_1, & q_1, & r_1 \\ 0, & 0, & 0, & p_2, & q_2, & r_2 \\ 0, & 0, & 0, & p_3, & q_3, & r_3 \end{vmatrix},$$

their product is the determinant of the sixth order

$$\begin{vmatrix} a_1, & b_1, & c_1, & 0, & 0, & 0 \\ a_2, & b_2, & c_2, & 0, & 0, & 0 \\ a_3, & b_3, & c_3, & 0, & 0, & 0 \\ 0, & 0, & 0, & p_1, & q_1, & r_1 \\ 0, & 0, & 0, & p_2, & q_2, & r_2 \\ 0, & 0, & 0, & p_3, & q_3, & r_3 \end{vmatrix}.$$

This rule is interesting as giving us a complete scale whereby we may represent the product of two determinants of order n by a determinant of any order from n to $2n$ inclusive; it is also frequently useful in applications of the theory.

8. The fundamental theorem of Art. 3 regarding the determinant formed by compounding two arrays can be deduced as follows from Laplace's theorem, IV. 5.

We can write the determinant $|c_{mm}|$ in the form of the determinant of order $(n+m)$, IV. 14,

$$\begin{vmatrix} c_{11} \dots c_{m1}, & b_{11} \dots b_{n1} \\ \dots\dots\dots & \dots\dots\dots \\ c_{1m} \dots c_{mm}, & b_{1m} \dots b_{nm} \\ 0 \;\dots\; 0, & 1 \;\dots\; 0 \\ \dots\dots\dots & \dots\dots\dots \\ 0 \;\dots\; 0, & 0 \;\dots\; 1 \end{vmatrix},$$

where c_{ik} has the value ascribed to it in Art. 2.

Now from the ith column subtract the last n columns multiplied respectively by $a_{i1}, a_{i2} \dots$ then from the value of c_{ik} it follows that

$$|c_{mm}| = \begin{vmatrix} 0 \;\dots\; 0, & b_{11} \dots b_{n1} \\ \dots\dots\dots & \dots\dots\dots \\ 0 \;\dots\; 0, & b_{1m} \dots b_{nm} \\ -a_{11} \dots -a_{m1}, & 1 \;\dots\; 0 \\ \dots\dots\dots & \dots\dots\dots \\ -a_{1n} \dots -a_{mn}, & 0 \;\dots\; 1 \end{vmatrix}.$$

In the determinant on the right multiply the first m columns

by -1 and then move the second m rows to the beginning, then (after $m+m^2$ changes of sign) our determinant is equal to

$$\begin{vmatrix} a_{11} & \dots & a_{m1}, & 1 & \dots & 0, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1m} & \dots & a_{mm}, & 0 & \dots & 1, & 0 & \dots & 0 \\ 0 & \dots & 0, & b_{11} & \dots & b_{m1}, & b_{m+1,1} & \dots & b_{n1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0, & b_{1m} & \dots & b_{mm}, & b_{m+1,m} & \dots & b_{nm} \\ a_{1,m+1} & \dots & a_{m,m+1}, & 0 & \dots & 0, & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_{1n} & \dots & a_{mn}, & 0 & \dots & 0, & 0 & \dots & 1 \end{vmatrix}.$$

Now expand this by Laplace's theorem according to minors of the first m columns. Let us find the complement of the minor

$$\begin{vmatrix} a_{1f}, & a_{2f} \dots \\ a_{1g}, & a_{2g} \dots \\ \dots & \dots \end{vmatrix}.$$

For this purpose we move the rows of a's having the suffixes $f, g \dots$ up to the beginning; then move those columns of b's which have the suffixes $f, g \dots$ into the $(m+1)^{\text{st}}, (m+2)^{\text{nd}} \dots$ places. This does not alter the value or sign of the determinant, and in every place where a 1 stood before, will now again stand 1. Hence the required complement is

$$\begin{vmatrix} b_{f1}, & b_{g1} \dots & 0 & 0 \\ b_{f2}, & b_{g2} \dots & 0 & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 \dots & 1 & 0 \\ 0 & 0 \dots & 0 & 1 \end{vmatrix} = \begin{vmatrix} b_{f1}, & b_{g1} \dots \\ b_{f2}, & b_{g2} \dots \\ \dots & \dots \end{vmatrix}.$$

Hence

$$|c_{mm}| = \Sigma \begin{vmatrix} a_{1f}, & a_{2f} \dots \\ a_{1g}, & a_{2g} \dots \\ \dots & \dots \end{vmatrix} . \begin{vmatrix} b_{f1}, & b_{g1} \dots \\ b_{f2}, & b_{g2} \dots \\ \dots & \dots \end{vmatrix}$$

where $f, g \dots$ is an m-ad from $1, 2 \dots n$. This agrees with our former result.

9. The value of any minor of order μ of the determinant $|c_{nn}|$, the product of two determinants $|a_{nn}|$ and $|b_{nn}|$, obtained

from $(a_{nn})\,(b_{nn})'$, say

$$C_\mu = \begin{vmatrix} c_{fp}, & c_{fq} & \dots & c_{fs} \\ c_{gp}, & c_{gq} & \dots & c_{gs} \\ \dots & \dots & \dots & \dots \\ c_{kp}, & c_{kq} & \dots & c_{ks} \end{vmatrix},$$

can be expressed as the sum of products of corresponding minors of order μ of the determinants $|a_{nn}|$ and $|b_{nn}|$.

In fact, since $\mu < n$, it follows at once from case (iii) of Art. 3 that

$$C_\mu = \Sigma \begin{vmatrix} a_{fi}, & a_{fj} & \dots & a_{fr} \\ a_{gi}, & a_{gj} & \dots & a_{gr} \\ \dots & \dots & \dots & \dots \end{vmatrix} \begin{vmatrix} b_{pi}, & b_{pj} & \dots & b_{pr} \\ b_{qi}, & b_{qj} & \dots & b_{qr} \\ \dots & \dots & \dots & \dots \end{vmatrix}$$

where $i, j \dots r$ is any μ-ad from $1, 2 \dots n$.

One particular case of this we shall find presently of importance; namely, when the two systems a and b are identical, and when moreover $f = p,\ g = q, \dots k = s$, so that the leading diagonal of C_μ consists of elements from the leading diagonal of $|c_{nn}|$.

Then we see that

$$C_\mu = \Sigma \begin{vmatrix} a_{fi}, & a_{fj} & \dots & a_{fr} \\ a_{gi}, & a_{gj} & \dots & a_{gr} \\ \dots & \dots & \dots & \dots \end{vmatrix}^2$$

a sum of n_μ squares.

10. The differential coefficients of a determinant C, elements c_{ik}, which is the product (effected by rows) of two determinants A, B, elements a_{ik}, b_{ik}, can be represented as the sum of products of differential coefficients of these determinants.

We have $$AB = C \quad\dots\dots\dots\dots(1),$$

and $$c_{ik} = a_{i1}b_{k1} + a_{i2}b_{k2} + \dots + a_{in}b_{kn}.$$

Differentiate (1) with regard to a_{ip}; remembering that $c_{i1}, c_{i2} \dots c_{in}$ are functions of this, we get

$$B\frac{\partial A}{\partial a_{ip}} = \frac{\partial C}{\partial c_{i1}} b_{1p} + \frac{\partial C}{\partial c_{i2}} b_{2p} + \dots + \frac{\partial C}{\partial c_{in}} b_{np}.$$

Multiply this equation by

$$\frac{\partial B}{\partial b_{kp}} = B_{kp}$$

and add together all the equations which can be obtained from it by writing for p the values $1, 2 \dots n$. Thus we get

$$B\Sigma \frac{\partial A}{\partial a_{ip}} \frac{\partial B}{\partial b_{kp}} = \frac{\partial C}{\partial c_{i1}} \Sigma B_{kp} b_{1p} + \dots + \frac{\partial C}{\partial c_{in}} \Sigma B_{kp} b_{np}.$$

But by IV. 11 all the sums on the right vanish except $\Sigma B_{kp} b_{kp}$, which is equal to B, hence

$$\frac{\partial C}{\partial c_{ik}} = \Sigma \frac{\partial A}{\partial a_{ip}} \frac{\partial B}{\partial b_{kp}} \quad (p = 1, 2 \dots n).$$

Similarly we can prove the equations

$$\frac{\partial^2 C}{\partial c_{ik} \partial c_{rs}} = \frac{1}{1 \,.\, 2} \Sigma \frac{\partial^2 A}{\partial a_{ip} \partial a_{rq}} \frac{\partial^2 B}{\partial b_{kp} \partial b_{sq}} \quad (p, q = 1, 2 \dots n),$$

$$\frac{\partial^3 C}{\partial c_{ik} \partial c_{pq} \partial c_{rs}} = \frac{1}{1 \,.\, 2 \,.\, 3} \Sigma \frac{\partial^3 A}{\partial a_{iu} \partial a_{pv} \partial a_{rw}} \frac{\partial^3 B}{\partial b_{ku} \partial b_{qv} \partial b_{sw}} \qquad (u, v, w = 1, 2 \dots n),$$

whence the general law is obvious.

CHAPTER VI.

ON DETERMINANTS OF COMPOUND SYSTEMS.

1. If the elements of a determinant are not simple quantities but themselves determinants, the determinant is called a compound determinant.

Compound determinants are usually formed from the minors of one or more determinants.

2. The number of all possible minors of order m of a given determinant of order n is $\{n_m\}^2$ (IV. 1). We can form a square array with these minors, writing in the same row all those which proceed from the same selection of rows of the given determinant, and similarly for the columns.

If $n_m = \mu$ and we give to the combinations of rows and columns taken to form minors the suffixes $1, 2 \dots \mu$, we may denote that minor whose elements belong to the ith combination of rows and jth combination of columns, by p_{ij}, and the whole system of minors will be

$$\left.\begin{matrix} p_{11} \cdots p_{1\mu} \\ \cdots\cdots\cdots\cdots \\ p_{\mu 1} \cdots p_{\mu\mu} \end{matrix}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (1).$$

Corresponding to each element in this array, which is a minor of the original determinant, we have a complementary minor of order $n-m$. We shall denote the complement of p_{ij} by q_{ij}, then these form a new array,.

$$\left.\begin{matrix} q_{11} \cdots q_{1\mu} \\ \cdots\cdots\cdots\cdots \\ q_{\mu 1} \cdots q_{\mu\mu} \end{matrix}\right\} \dots\dots\dots\dots\dots\dots\dots\dots (2).$$

The arrays (1) and (2) are called reciprocal arrays of the mth order. Minors of these arrays formed from the same selection of rows and columns in each are called conjugate minors. The simplest instance of two such arrays is the original system and its system of first minors, viz.

$$\begin{array}{lcl} a_{11} \dots a_{1n} & \qquad & A_{11} \dots A_{1n} \\ \dots\dots\dots & & \dots\dots\dots \\ a_{n1} \dots a_{nn} & & A_{n1} \dots A_{nn}. \end{array}$$

3. If we multiply the elements of the ith row of the array (1) by the corresponding elements of the kth row of (2) the sum of the products is equal to A or zero according as i is or is not equal to k, viz.

$$p_{i1}q_{k1} + p_{i2}q_{k2} + \dots + p_{i\mu}q_{k\mu} = \delta_{ik}A.$$

For if i is equal to k this is nothing else than the expansion of the given determinant A according to products of minors of order m and $n-m$ by Laplace's theorem. If i is not equal to k the sum represents the expansion of the determinant when the ith selection of rows is replaced by the kth; the rows of this determinant are not all different, hence it vanishes. The particular case

$$a_{i1}A_{k1} + a_{i2}A_{k2} + \dots + a_{in}A_{kn} = \delta_{ik}A$$

is considered in IV. 11.

4. We now proceed to investigate properties of determinants of the elements of reciprocal systems, and first we shall examine the system of the first order.

Let $\qquad A = |a_{nn}|, \qquad D = |A_{nn}|.$

Forming the product of these two by rows,

$$AD = |C_{nn}|,$$

where $\qquad C_{ik} = a_{i1}A_{k1} + a_{i2}A_{k2} + \dots + a_{in}A_{kn},$

and hence $C_{ik} = A$ or 0 according as i is or is not equal to k.

Thus
$$AD = \begin{vmatrix} A, & 0, & 0 \dots \\ 0, & A, & 0 \dots \\ 0, & 0, & A \dots \\ \dots & \dots & \dots \end{vmatrix} = A^n;$$

$$\therefore \; D = A^{n-1}.$$

5. Any minor of order p in the system A_{ik} is equal to the complementary minor of its conjugate in A multiplied by A^{p-1}.

Let
$$\Sigma \pm a_{fi} a_{gk} \dots = \begin{vmatrix} a_{fi}, & a_{fk} \dots \\ a_{gi}, & a_{gk} \dots \\ \dots & \dots \end{vmatrix}$$

and $\Sigma \pm A_{fi} A_{gk} \dots$ be two conjugate minors in the two systems each of order p, and let $\Sigma \pm a_{ru} a_{sv} \dots$ be the complement of $\Sigma \pm a_{fi} a_{gk} \dots$. Then, if $\epsilon = 1$ or -1 according to circumstances,

$$\epsilon A = \begin{vmatrix} a_{fi}, & a_{fk} \dots a_{fu}, & a_{fv} \dots \\ a_{gi}, & a_{gk} \dots a_{gu}, & a_{gv} \dots \\ \dots & \dots & \dots \\ a_{ri}, & a_{rk} \dots a_{ru}, & a_{rv} \dots \\ a_{si}, & a_{sk} \dots a_{su}, & a_{sv} \dots \\ \dots & \dots & \dots \end{vmatrix} = \Sigma \pm a_{fi} a_{gk} \dots a_{ru} a_{sv} \dots \qquad (1).$$

We may write $\Sigma \pm a_{ru} a_{sv} \dots = \mathrm{co}\, \Sigma \pm a_{fi} a_{gk} \dots$.

Now we may write $\Sigma \pm A_{fi} A_{gk} \dots$ as the determinant of order n,

$$\begin{vmatrix} A_{fi}, & A_{fk} \dots A_{fu}, & A_{fv} \dots \\ A_{gi}, & A_{gk} \dots A_{gu}, & A_{gv} \dots \\ \dots & \dots & \dots \\ 0, & 0 \quad \dots 1, & 0 \quad \dots \\ 0, & 0 \quad \dots 0, & 1 \quad \dots \\ \dots & \dots & \dots \end{vmatrix},$$

which consists of four parts. The first square consists of the elements of $\Sigma \pm A_{fi} A_{gk} \dots$; to the right of this is a rectangle of $n-p$ columns and p rows containing the remaining elements of the fth, gth ... rows of $|A_{nn}|$. The rectangle on the left below of p columns and $n-p$ rows consists solely of zeros, and the

square on the right of $n-p$ rows and columns contains units in the leading diagonal and zeros elsewhere. Multiply this by the determinant A written in the form (1) above. Then (IV. 11) we have

$$\epsilon A \Sigma \pm A_{fi} A_{gk} \ldots = \begin{vmatrix} A, & 0 \ldots a_{fu}, & a_{fv} \ldots \\ 0, & A \ldots a_{gu}, & a_{gv} \ldots \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \\ 0, & 0 \ldots a_{ru}, & a_{rv} \ldots \\ 0, & 0 \ldots a_{su}, & a_{sv} \ldots \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots} \end{vmatrix}$$

$$= A^p \Sigma \pm a_{ru} a_{sv} \ldots$$

if we resolve the determinant on the right into products of minors of the first p and last $n-p$ columns. Accordingly

$$\Sigma \pm A_{fi} A_{gk} \ldots = A^{p-1} \operatorname{co} \Sigma \pm a_{fi} a_{gk} \ldots$$

where it is unnecessary to retain the factor ϵ, provided that we make a suitable convention as to the signs of the determinants on each side (cf. IV. 10).

From this it follows that the ratio of two minors of the same order of the system A_{ik} is the same as the ratio of the complementary minors of their conjugates,

$$\frac{\Sigma \pm A_{fi} A_{gh} \ldots}{\Sigma \pm A_{kl} A_{pq} \ldots} = \frac{\operatorname{co} \Sigma \pm a_{fi} a_{gh} \ldots}{\operatorname{co} \Sigma \pm a_{kl} a_{pq} \ldots}.$$

6. As examples of the theorem in Art. 5, we have

$$\begin{vmatrix} A_{11} \ldots A_{1p} \\ \ldots\ldots\ldots\ldots \\ A_{p1} \ldots A_{pp} \end{vmatrix} = A^{p-1} \begin{vmatrix} a_{p+1,\,p+1} \ldots a_{p+1,\,n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n,\,p+1} \quad \ldots a_{nn} \end{vmatrix},$$

$$\begin{vmatrix} A_{p+1,\,p+1} \ldots A_{p+1,\,n} \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ A_{n,p+1} \quad \ldots A_{nn} \end{vmatrix} = A^{n-p-1} \begin{vmatrix} a_{11} \ldots a_{1p} \\ \ldots\ldots\ldots \\ a_{p1} \ldots a_{pp} \end{vmatrix}.$$

The relation

$$\begin{vmatrix} A_{ik}, & A_{is} \\ A_{rk}, & A_{rs} \end{vmatrix} = A \operatorname{co} \begin{vmatrix} a_{ik}, & a_{is} \\ a_{rk}, & a_{rs} \end{vmatrix}$$

may also be written

$$\frac{\partial A}{\partial a_{ik}}\frac{\partial A}{\partial a_{rs}}-\frac{\partial A}{\partial a_{is}}\frac{\partial A}{\partial a_{rk}}=A\frac{\partial^2 A}{\partial a_{ik}\partial a_{rs}},$$

in particular

$$\frac{\partial A}{\partial a_{n-1,\,n-1}}\frac{\partial A}{\partial a_{nn}}-\frac{\partial A}{\partial a_{n-1,\,n}}\frac{\partial A}{\partial a_{n,\,n-1}}=A\frac{\partial^2 A}{\partial a_{n-1,\,n-1}\partial a_{nn}}.$$

If $A=0$, we see that

$$\begin{vmatrix} A_{ik}, & A_{is} \\ A_{rk}, & A_{rs} \end{vmatrix}=0,$$

or

$$\frac{A_{ik}}{A_{rk}}=\frac{A_{is}}{A_{rs}}.$$

That is to say, if the determinant vanishes, the minors of the elements of any row are proportional to the corresponding minors of the elements of any other row.

7. As an example of the use of the method of Arts. 20 and 21 of Chap. IV., let us discuss the value of the determinant

$$P=|\lambda a_{ik}+\mu b_{ik}|,$$

a_{ik} and b_{ik} being elements of two determinants of the nth order

$$A^{(n)}=|a_{ik}|,\qquad B^{(n)}=|b_{ik}|.$$

Symbolically we can write

$$P=(\lambda A+\mu B)^n$$

$$=A^nB^n\left(\frac{\lambda}{B}+\frac{\mu}{A}\right)^n.$$

Now let $A_1{}^{(n)}$, $B_1{}^{(n)}$ be two determinants of order n, whose elements are

$$\alpha_{ik}=\frac{1}{A^{(n)}}\frac{\partial A^{(n)}}{\partial a_{ik}},\qquad \beta_{ik}=\frac{1}{B^{(n)}}\frac{\partial B^{(n)}}{\partial b_{ik}},$$

then by Art. 4

$$A_1{}^{(n)}=\frac{|A_{ik}{}^{(n)}|}{(A^{(n)})^n}=\frac{1}{A^{(n)}},$$

and similarly

$$B_1{}^{(n)}=\frac{1}{B^{(n)}}.$$

Or, symbolically,

$$A_1 = \frac{1}{A}, \quad B_1 = \frac{1}{B}.$$

Thus $$P = A^n B^n (\lambda B_1 + \mu A_1)^n.$$

But $(\lambda B_1 + \mu A_1)^n$ is the symbolical expression for a determinant of order n with binomial elements of the form

$$\lambda\beta_{ik} + \mu\alpha_{ik}.$$

Hence, passing from symbolic to real expressions, we have the determinant equation:

$$|\lambda a_{ik} + \mu b_{ik}| = |a_{ik}| \,.\, |b_{ik}| \,.\, |\lambda\beta_{ik} + \mu\alpha_{ik}|.$$

Numerous other transformations of the determinant on the left can be effected.

8. Next let us consider reciprocal arrays of order m. With the notation of Art. 2, let

$$\Delta = |p_{\mu\mu}|, \quad \Delta' = |q_{\mu\mu}|,$$

where, as before, $\mu = n_m$.

The product $\Delta\Delta'$ is a determinant of order μ whose general element is

$$p_{i1}q_{k1} + p_{i2}q_{k2} + \ldots + p_{i\mu}q_{k\mu},$$

which is equal to A or 0 according as i is or is not equal to k. (Art. 3.) Hence in the product determinant all the elements vanish except those in the principal diagonal.

Thus $$\Delta\Delta' = A^\mu.$$

It follows therefore that Δ is a divisor of A^μ. Now A is a linear function of any one of its elements, hence Δ can only differ from a power of A by a coefficient independent of the elements of A. Among the combinations m at a time of the numbers $1, 2 \ldots n$ there are

$$\lambda = (n-1)_{m-1},$$

which contain 1. Hence there are λ elements of Δ, which contain a_{11}, and consequently

$$\Delta = xA^\lambda,$$

where x does not depend on the elements of A.

To determine the value of x, let $a_{ik}=0$ except when $i=k$, and let $a_{ii}=1$. The same will be the case with the elements p_{ik};

$$\therefore\ A=1, \quad \Delta=1, \quad \text{and } \therefore\ x=1.$$

Thus
$$\Delta = A^{(n-1)_{m-1}},$$
and
$$\Delta' = A^{(n-1)_m}$$
for
$$n_m-(n-1)_{m-1}=(n-1)_m.$$

9. A minor of order r of the system q_{ik} is equal to the complement of its conjugate multiplied by $A^{r-\lambda}$.

For if we multiply the determinant $\Sigma \pm q_{fi}q_{gk}\ldots$ by the determinant Δ in the same manner as we did in Art. 5 for systems of the first order, we get:

$$\Delta\Sigma \pm q_{fi}q_{gk}\ldots = A^r \operatorname{co}\Sigma \pm p_{fi}p_{gk}\ldots;$$

and therefore, since $\Delta = A^\lambda$,

$$\Sigma \pm q_{fi}q_{gk}\ldots = A^{r-\lambda}\operatorname{co}\Sigma \pm p_{fi}p_{gk}\ldots = A^{r-(n-1)_{m-1}}\operatorname{co}\Sigma \pm p_{fi}p_{gk}\ldots.$$

In like manner

$$\Sigma \pm p_{fi}p_{gk}\ldots = A^{r-(n-1)_m}\operatorname{co}\Sigma \pm q_{fi}q_{gk}\ldots.$$

10. Let A_h be a minor of A, with h rows and columns. From this let us form the determinant whose elements are all the minors of order m of A_h. These last are minors of order m of A, and are consequently elements of Δ. Moreover, those among them which arise from the same rows or columns of A, and are hence in the same row or column of Δ, also arise from elements belonging to the same row or column of A_h, which is a minor of A; altogether they form a minor M of Δ, which has h_m rows and columns. Now by Art. 8 we have

$$M = A_h^{(h-1)_{m-1}},$$

which gives a representation of minors of Δ by means of powers of minors of A.

11. If in the determinant A we select a minor A_h of order h, and form all the minors of order m in A $(m>h)$, which contain neither all the h rows nor all the h columns of A_h, we shall form a minor of Δ with $n_m-(n-h)_{m-h}$ rows and columns, which is equal to

$$A_{n-h}^{(n-h-1)_{m-h}} \cdot A^{\{(n-1)_{m-1}-(n-h)_{m-h}\}},$$

where A_{n-h} is the complement of A_h in A.

To prove this, we begin by applying the result of Art. 10, with the substitution of A_{n-h} for A_h, $(n-m)$ for m, and Δ' for Δ. Instead of M we now have a determinant M' which is a minor of Δ' containing

$$(n-h)_{n-m} = (n-h)_{m-h}$$

rows and columns. The value of this is (since $n-h > n-m$)

$$M' = A_{n-h}^{(n-h-1)_{n-m-1}} = A_{n-h}^{(n-h-1)_{m-h}}.$$

Now let us consider α_1, that minor of Δ whose elements are the complementary minors in A of the elements of M'. Since M' has for its elements all the minors of A_{n-h} which are of order $(n-m)$ it follows that the elements of α_1 are all the minors of A of order m which have A_h as a minor. The order of α_1 is $(n-h)_{m-h}$, and hence by Art. 9, if α is the complement of α_1 in Δ,

$$\begin{aligned} \alpha &= M' \,.\, A^{(n-1)_{m-1}-(n-h)_{m-h}} \\ &= A_{n-h}^{(n-h-1)_{m-h}} \,.\, A^{(n-1)_{m-1}-(n-h)_{m-h}}. \end{aligned}$$

The theorem is therefore proved, if we can shew that α is formed as prescribed. For this purpose we must remember that α_1 has for elements all minors (of order m) of A which have A_h for one of their minors; to get α we have then to suppress among the combinations m at a time of the rows and columns of A all those which contain all the rows or columns of A_h; thus α has for its elements all the minors of A with m rows and columns, such that they do not contain all the h rows or columns of A_h.

12. If A_h is a minor of order h of A, and if we border it in all possible ways with m of the remaining rows and columns of A, we get the elements of a new determinant M_m of order $(n-h)_m$, whose value is

$$A_h^{(n-h-1)_m} \,.\, A^{(n-h-1)_{m-1}}.$$

Let us put, for the moment, $n = h + k$, and write A and its reciprocal in the abbreviated forms

$$A = \begin{vmatrix} (a_{hh}) & (b_{hk}) \\ (c_{kh}) & (d_{kk}) \end{vmatrix}, \quad R = \begin{vmatrix} (A_{hh}) & (B_{hk}) \\ (C_{kh}) & (D_{kk}) \end{vmatrix},$$

where $|a_{hh}| = A_h$.

Consider the determinant of order k_m whose elements are all the minors of order $(k-m)$ of $|D_{kk}|$. It follows from Art. 8 that its value is

$$|D_{kk}|^{(k-1)_{k-m-1}} = (A^{k-1}A_h)^{(k-1)_m},$$

since, by Art. 5, $|D_{kk}| = A^{k-1}A_h$, and $(k-1)_{k-m-1} = (k-1)_m$.

But any element β_{ij} of the determinant, as being a minor of R of order $(k-m)$, can be expressed by Art. 5 in the form

$$\beta_{ij} = A^{k-m-1}\alpha_{ij},$$

where α_{ij} is an element of M_m. Consequently (putting $k_m = \lambda$)

$$|\beta_{\lambda\lambda}| = M_m A^{(k-m-1)k_m}.$$

Equating this to the value of $|\beta_{\lambda\lambda}|$ previously obtained, and observing that

$$(k-1)(k-1)_m - (k-m-1)\,k_m = (k-1)_{m-1},$$

we find that

$$M_m = A_h^{(k-1)_m}\,.\,A^{(k-1)_{m-1}},$$

which is equivalent to the theorem stated at the beginning of this article.

13. Another way of stating the proposition is the following: If A_h is a minor of order h of A, and we form all the minors of A with m rows and columns which have it as a minor, we get the elements of a new determinant of order $(n-h)_{m-h}$, whose value is

$$A_h^{(n-h-1)_{m-h}}\,.\,A^{(n-h-1)_{m-h-1}}.$$

In the particular case of $m = h+1$, the theorem is

$$M_1 = A_h^{n-h-1}\,.\,A,$$

and the elements of M_1 are determinants obtained from A_h by adding one extra row and one extra column: or, which is the same thing, they are those minors of A which have A_h for a first minor.

The results of this article are due to Sylvester.

14. Another modification of the theorem of Art. 12 can be obtained as follows: Let us return to the determinants Δ, Δ' of Art. 8, and form a determinant M' with the minors of A_{n-h} of order $n-m$; this is a minor of Δ' of order $(n-h)_{m-h}$.

The conjugate minor in Δ has for elements those minors of A of order m which are complementary to the elements of M', and hence all those which have A_h as a minor. This is precisely the determinant of Art. 13. Whence the theorem can be stated as follows: If A_{n-h} is a minor of A of order $n-h$, and if we form a determinant M' with all the minors of order $n-m$ of A_{n-h}, and then replace each element by its complement in A, we get a new determinant, whose value is

$$M = A_h^{(n-h-1)_{m-h}} \,.\, A^{(n-h-1)_{m-h-1}}.$$

15. If now we form all minors of A of order $n-m\,(m>h)$ such that neither all their rows nor all their columns belong to A_{n-h}, which in A therefore overlap A_{n-h} or belong altogether to A_h, these form a determinant N of order $n_m-(n-h)_{m-h}$ which is equal to

$$A_h^{(n-h-1)_{m-h}} \,.\, A^{(n-1)_m-(n-h-1)_{m-h}}.$$

First notice that this is essentially different from the theorem of Art. 11, applied to A_h. There the determinant is formed with all the minors of the same order of A with more elements than A_h, and which do not admit all the rows and columns of A_h. Here the determinant is formed with minors of the same order of A with fewer elements than A_{n-h}, and which do not admit all the rows and columns A_{n-h}.

To prove the theorem it is sufficient to consider in Δ' the minor N complementary to M in Δ'. For N is exactly formed with regard to A_{n-h} as the enunciation prescribes; it has $n_m-(n-h)_{m-h}$ rows, and by applying to it the theorem of Art. 9,

$$\frac{M}{N} = A^{(n-1)_{m-1}-n_m+(n-h)_{m-h}},$$

$$\therefore\quad N = MA^{(n-1)_m-(n-h)_{m-h}};$$

or, replacing M by its value, from Art. 14,

$$N = A_h^{(n-h-1)_{m-h}} \,.\, A^{(n-1)_m-(n-h-1)_{m-h}}.$$

16. The theorem of Art. 15 may be stated in a different way, which perhaps brings out more clearly its contrast with the proposition of Art. 11.

Suppose as before that A_h is a selected minor of A, of order h.

Let $m < h$, and construct all the minors of A which are of order m and contain at least one row and at least one column not composed of elements of A_h. With these minors as elements we can form a determinant of order $n_m - h_m$, the value of which is

$$A_{n-h}^{(h-1)_{m-1}} \cdot A^{(n-1)_{m-1}-(h-1)_{m-1}}.$$

This is at once obtained from the result of Art. 15 by substituting $n-m$ for m and $n-h$ for h, and observing that

$$(h-1)_{h-m} = (h-1)_{m-1}, \qquad (n-1)_{n-m} = (n-1)_{m-1}.$$

17. Netto has given a theorem which may be regarded as an extension of Laplace's formula (IV. 5) for the expansion of a determinant as the sum of products of minors.

Let a determinant of order $(n+m)$ and its reciprocal be represented by

$$A = \begin{vmatrix} (a_{nn}) & (b_{nm}) \\ (c_{mn}) & (d_{mm}) \end{vmatrix}, \quad R = \begin{vmatrix} (A_{nn}) & (B_{nm}) \\ (C_{mn}) & (D_{mm}) \end{vmatrix},$$

and let $|a_{nn}|$ be denoted by A_n.

Let k integers $m_1, m_2, \ldots m_k$ be chosen so that

$$m_1 + m_2 + \ldots + m_k = n.$$

If we choose minors of A_n of orders $m_1, m_2, \ldots m_k$ formed out of elements in the first m_1 rows, the next m_2 rows, etc. and denote them by $\delta_{m_1}, \delta_{m_2}, \ldots \delta_{m_k}$, Laplace's theorem gives

$$A_n = \Sigma \pm \delta_{m_1} \delta_{m_2} \ldots \delta_{m_k}.$$

Now δ_{m_i} is obtained from A_n by the suppression of $(n-m_i)$ rows and $(n-m_i)$ columns; let Δ_{m_i} be the minor of A obtained by the suppression of the same rows and columns. Then the theorem in question is that

$$\Sigma \pm \Delta_{m_1} \Delta_{m_2} \ldots \Delta_{m_k} = A D_m^{k-1},$$

where $D_m = |d_{mm}|$, the complement of A_n in A.

To prove this, let us, in the first place, apply Laplace's theorem to expand the reciprocal of A_n. Thus

$$A_n^{n-1} = \Sigma \pm \delta'_{m_1} \delta'_{m_2} \ldots \delta'_{m_k},$$

where δ'_{m_i} is the conjugate of δ_{m_i} in the reciprocal of A_n. But if δ_{n-m_i} is the complement of δ_{m_i} in A_n

$$\delta'_{m_i} = \delta_{n-m_i} A_n{}^{m_i-1};$$

consequently

$$A_n{}^{n-1} = A_n{}^{\Sigma(m_i-1)} \Sigma \pm \delta_{n-m_1}\delta_{n-m_2}\ldots\delta_{n-m_k},$$

whence

$$\Sigma \pm \delta_{n-m_1}\delta_{n-m_2}\ldots\delta_{n-m_k} = A_n{}^{k-1}.$$

Next we observe that by Art. 5

$$\Delta_{m_i} A^{n-m_i-1} = \Delta'_{n-m_i},$$

the complement in R of the conjugate of Δ_{m_i}. Now this is a minor of $|A_{nn}|$ and by applying the preceding lemma to $|A_{nn}|$ instead of to A_n we obtain

$$\Sigma \pm \Delta'_{n-m_1}\Delta'_{n-m_2}\ldots\Delta'_{n-m_k} = |A_{nn}|^{k-1} = A^{(n-1)(k-1)} D_m{}^{k-1}.$$

Substituting for Δ'_{n-m_i} its value given above, and making use of the relation

$$\Sigma(n-m_i-1) = k(n-1) - n = (n-1)(k-1) - 1,$$

we have finally

$$\Sigma \pm \Delta_{m_1}\Delta_{m_2}\ldots\Delta_{m_k} = A D_m{}^{k-1}$$

as stated. This proof of Netto's theorem is due to Pascal.

18. The theorem of Art. 13, in its simplest case, may be used to prove an important identity discovered by Kronecker.

Suppose we have a determinant of order $2h+1$

$$\begin{vmatrix} (a_{hh}) & (b_{h,\,h+1}) \\ (c_{h+1,\,h}) & (d_{h+1,\,h+1}) \end{vmatrix} = D,$$

the elements of which are all independent.

Let $|a_{hh}| = A$, and let $A_{\lambda\mu}$ be the determinant of order $(h+1)$ obtained by bordering A with elements taken from the row and column to which $d_{\lambda\mu}$ belongs. That is to say,

$$A_{\lambda\mu} = \begin{vmatrix} a_{11} & a_{12} \ldots a_{1h} & b_{1\mu} \\ \cdots\cdots & \cdots\cdots & \cdots \\ a_{h1} & a_{h2} \ldots a_{hh} & b_{h\mu} \\ c_{\lambda 1} & c_{\lambda 2} \ldots c_{\lambda h} & d_{\lambda\mu} \end{vmatrix}.$$

Then

$$\begin{vmatrix} A_{11} - d_{11} A, & A_{12} - d_{12} A, & \ldots A_{1,h+1} - d_{1,h+1} A \\ \vdots & \vdots & \vdots \\ A_{h+1,1} - d_{h+1,1} A, & A_{h+1,2} - d_{h+1,2} A, & \ldots A_{h+1,h+1} - d_{h+1,h+1} A \end{vmatrix} = 0$$

identically.

The proof is almost immediate. By Sylvester's theorem

$$|A_{h+1,h+1}| = DA^h;$$

now suppose all the elements d_{ik} to become zero, then D vanishes, as we see by expanding according to minors of the last $h+1$ rows. But since $A_{ik} - d_{ik} A$ is independent of the coefficients d_{ik} this expression is in fact the value of A_{ik} when all the coefficients d_{ik} vanish. Hence

$$|A_{ik} - d_{ik} A| = 0, \qquad (i, k = 1, 2, \ldots h+1)$$

and this is an identity on account of the independence of the $(3h^2 + 2h)$ quantities which it involves. The method of proof here followed is due to Frobenius.

19. There are various interesting theorems about determinants derived from two other independent determinants. One of these, due to Kronecker, is the following.

Let $A = |a_{nn}|$, $B = |b_{mm}|$, and let us form all the m^2n^2 products

$$a_{ij} b_{hk};$$

then with these products as elements we can construct a determinant

$$C = |c_{\mu\mu}| \qquad (\mu = mn)$$

of order mn, in such a way that all the elements $a_{ij} b_{hk}$ with constant i, h are in the same row, and all those with constant j, k are in the same column. This being so, the value of C is

$$C = A^m B^n.$$

Without loss of generality we may suppose C to have the form

$$\begin{vmatrix} a_{11}\,(b_{mm}) & a_{12}\,(b_{mm}) & \ldots\, a_{1n}\,(b_{mm}) \\ a_{21}\,(b_{mm}) & a_{22}\,(b_{mm}) & \ldots\, a_{2n}\,(b_{mm}) \\ \cdots & \cdots & \cdots \\ a_{n1}\,(b_{mm}) & a_{n2}\,(b_{mm}) & \ldots\, a_{nn}\,(b_{mm}) \end{vmatrix}.$$

If, now, we form minors of order m out of the first m rows of C, all those which do not vanish will contain B as a factor. Thus, for instance, the minor formed from $a_{11}(b_{mm})$ is $a_{11}{}^m B$; if we replace the first column by the first column of $a_{12}(b_{mm})$, we get $a_{11}{}^{m-1}a_{12}B$, and so on. The same argument applies to minors taken from the next m rows, etc.

By the generalised form of Laplace's theorem (IV. 8) we may expand C as a sum of products of n-ads of minors of order m chosen from the first, second, ... sets of m rows. Since each of the minors vanishes or is divisible by B it follows that C is divisible by B^n. In exactly the same way C is divisible by A^m, and the theorem now follows by a comparison of dimensions, and the consideration of the case when $a_{ii} = b_{ii} = 1$ and all the other elements are zero.

20. We shall now suppose that we have two independent determinants, each of order n,

$$A = |a_{nn}|, \qquad B = |b_{nn}|,$$

and that for each of them we have formed the systems of $\mu^2 (= n_m{}^2)$ elements discussed in Art. 2; the systems for A being denoted by $(p_{\mu\mu})$, $(q_{\mu\mu})$ and those for B by $(p'_{\mu\mu})$, $(q'_{\mu\mu})$.

We can form two new systems each of μ^2 elements as follows. If i is a fixed integer the quantities p'_{ik} are minors chosen from a particular set of m rows in B. In the determinant A replace, in turn, each selection of rows m at a time by this fixed selection of rows in B. This will give us μ determinants, each of order n, which we shall denote by $t_{i1}, t_{i2}, \ldots t_{i\mu}$.

Again let k be a fixed integer, and in B let those rows from which the quantities p'_{kj} are derived be replaced in turn by each combination of m rows of A; the determinants thus obtained will be called $u_{k1}, u_{k2}, \ldots u_{k\mu}$. We have, then, two new systems

$$(t_{\mu\mu}), \quad (u_{\mu\mu}).$$

Then by Laplace's theorem

$$A = p_{h1}q_{h1} + p_{h2}q_{h2} + \ldots \qquad B = p'_{h1}q'_{h1} + p'_{h2}q'_{h2} + \ldots$$

$$t_{ik} = p'_{i1}q_{k1} + p'_{i2}q_{k2} + \ldots \qquad u_{lm} = p_{m1}q'_{l1} + p_{m2}q'_{l2} + \ldots,$$

whence by Art. 3

$$\begin{aligned} t_{i1}p_{11} + t_{i2}p_{21} + \ldots &= p'_{i1}A \\ t_{i1}p_{12} + t_{i2}p_{22} + \ldots &= p'_{i2}A \\ \ldots\ldots\ldots\ldots\ldots\ldots&\ldots\ldots, \end{aligned}$$

and therefore

$$t_{i1}u_{k1} + t_{i2}u_{k2} + \ldots = A\,(p'_{i1}q'_{k1} + p'_{i2}q'_{k2} + \ldots) = \delta_{ik}AB.$$

That is to say, by compounding the ith row of $(t_{\mu\mu})$ with the kth row of $(u_{\mu\mu})$, we get AB or 0 according as i is or is not equal to k.

21. Write now

$$T = |\,t_{\mu\mu}\,|, \qquad U = |\,u_{\mu\mu}\,|;$$

then, by the result of Art. 20,

$$\begin{aligned} T &= |\,p'_{\mu\mu}\,|\,.\,|\,q_{\mu\mu}\,| = A^{(n-1)_m}\,.\,B^{(n-1)_{m-1}} \\ U &= |\,p_{\mu\mu}\,|\,.\,|\,q'_{\mu\mu}\,| = A^{(n-1)_{m-1}}\,.\,B^{(n-1)_m} \end{aligned}$$

(Art. 8).

Again let

$$\begin{aligned} T_h &= |\,t_{hh}\,| \\ &= \begin{vmatrix} t_{11} & t_{12} \ldots t_{1h} & t_{1,\,h+1} \ldots t_{1\mu} \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ t_{h1} & t_{h2} \ldots t_{hh} & t_{h,\,h+1} \ldots t_{h\mu} \\ 0 & 0 \ldots 0 & 1 \;\ldots\; 0 \\ \multicolumn{3}{c}{\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots} \\ 0 & 0 \ldots 0 & 0 \;\ldots\; 1 \end{vmatrix} \end{aligned}$$

then, forming the product T_hU by rows, and applying the results of Art. 20,

$$\begin{aligned} T_hU = (AB)^h &\begin{vmatrix} u_{h+1,\,h+1}, & u_{h+1,\,h+2} & \ldots & u_{h+1,\,\mu} \\ u_{h+2,\,h+1}, & u_{h+2,\,h+2} & \ldots & u_{h+2,\,\mu} \\ \ldots\ldots & \ldots\ldots & \ldots & \ldots\ldots \\ u_{\mu,\,h+1}, & u_{\mu,\,h+2} & \ldots & u_{\mu,\,\mu} \end{vmatrix} \\ = (AB)^h U_{\mu-h}&, \end{aligned}$$

where $U_{\mu-h}$ is the complement of U_h in U.

Substituting its value for U, we have

$$U_{\mu-h}/T_h = A^{\lambda_1}B^{\lambda_2},$$

where $\qquad \lambda_1 = (n-1)_{m-1} - h, \quad \lambda_2 = (n-1)_m - h.$

Thus the ratio of complementary minors of T and U is a power of A multiplied by a power of B.

CHAPTER VII.

ARITHMETICAL PROPERTIES OF DETERMINANTS. ELEMENTARY FACTORS.

1. A DETERMINANT is a rational integral function of its elements, and as such will have certain arithmetical properties dependent upon the field of rationality to which the elements belong.

If the elements are independent and arbitrary symbols, the determinant is irreducible, and so are all its minors. This is easily proved by induction; for the expansion

$$A = |a_{nn}| = a_{11}A_{11} + a_{12}A_{12} + \ldots + a_{1n}A_{1n}$$

shews that if A is reducible one of its factors must be a common divisor of $A_{11}, A_{12}, \ldots A_{1n}$. But these are determinants of order $(n-1)$, each with arbitrary elements, and distinct from each other; so that if the theorem is true for the order $(n-1)$, it is true for the order n. Since it is obviously true for $n=1$, it is true in general.

But the case is different when the elements belong to a field of a more special kind. For instance, if the elements are ordinary integers, the values of A and its minors are also integers, none of which need be prime. So also, if the elements are polynomials in x with integral coefficients, A and its minors are functions of the same type, and the nature and distribution of their irreducible (or prime) factors require special examination.

2. It will be supposed that in the determinant $|a_{nn}|$, or A, the elements are all integral and rational in a certain definite field; then the same will be true of A and all its minors. It will be further assumed that every integral quantity in the field can

be uniquely resolved into a product of prime integral factors. It follows from this that a given set of integral quantities possess a definite highest common factor, which can be found, as in ordinary arithmetic, when the resolution of the given quantities into prime factors has been effected.

For the present it will be supposed that the value of A is not zero: it follows that its minors of order $m\,(<n)$ cannot all simultaneously vanish.

3. Consider, now, all the minors of A which are of order σ. They will have a certain highest common factor; this will be denoted by D_σ. Altogether, we have n integral quantities

$$D_1,\ D_2,\ \dots\ D_n,$$

which we shall call the *determinant factors* of A. In particular, D_1 is the highest common factor of the elements of A, while D_n is A itself. The determinant factors are all different from zero.

We shall now prove that D_σ is divisible by $D_{\sigma-1}$.

This follows from the fact that, when any minor of order σ is expanded according to elements of one of its rows, each element is multiplied by a minor of order $(\sigma-1)$, and this is divisible by $D_{\sigma-1}$. We shall write

$$D_\sigma/D_{\sigma-1}=E_\sigma.$$

With the convention that $D_0=1$, we thus obtain a new set of integral quantities

$$E_1,\ E_2,\ \dots\ E_n,$$

which will be called the *elementary factors* of A. It will be noticed that $E_1=D_1$, and that

$$A=E_1E_2\dots E_n.$$

4. It is clear that no prime can divide D_σ which is not also contained in A. Let p^{l_σ} be the highest power of the prime factor p which is contained in D_σ. Then since D_σ is divisible by $D_{\sigma-1}$ it follows that

$$l_n \geqslant l_{n-1} \geqslant \dots \geqslant l_1,$$

so that if we write

$$e_\sigma = l_\sigma - l_{\sigma-1},$$

with the convention that $l_0=0$, we have the integers

$$e_1,\ e_2,\ e_3,\ \dots\ e_n,$$

none of which is negative, while

$$e_1 + e_2 + \dots + e_\sigma = l_\sigma.$$

In this notation

$$E_\sigma = \Pi p^{e_\sigma},$$

the product extending to all the different prime factors of A.

The values of D_σ and E_σ are not affected by interchange of rows or of columns in A; for this does not alter the values of the minors of order σ (except possibly in sign, which is immaterial) but merely produces a permutation amongst them.

5. If p is any prime factor of A there will be at least one minor of order σ which is divisible by p^{l_σ} but by no higher power of p. Such a minor is said to be *regular* with respect to p.

We shall now prove by induction the following three important propositions:—

(i) Every minor of order σ which is regular with respect to p has at least one first minor which is also regular.

(ii) Every regular minor of order $(\sigma - 1)$ is a first minor of at least one regular minor of order σ.

(iii) The indices e_i associated with the prime factor p satisfy the relation

$$e_\sigma \geqslant e_{\sigma-1}.$$

First of all we choose a non-vanishing minor of A which is of order $\rho\,(> 2)$; let this be

$$M = |a_{hi}| \qquad (h = h_1,\ h_2,\ \dots\ h_\rho;\ i = i_1,\ i_2,\ \dots\ i_\rho).$$

Let p^{λ_σ} be the highest power of p which is contained in all the minors of M which are of order σ. Choose a minor of M, of order $(\rho - 2)$, which is divisible by $p^{\lambda_{\rho-2}}$ but by no higher power; and let this minor be called T. Then (VI. 5, 6) we have an identity

$$MT = PS - QR,$$

where P, Q, R, S are minors of M of order $(\rho - 1)$. Now MT is divisible by $p^{\lambda_\rho + \lambda_{\rho-2}}$ and by no higher power of p, while $PS - QR$ is certainly divisible by $p^{2\lambda_{\rho-1}}$ and possibly by a higher power; hence

$$\lambda_\rho + \lambda_{\rho-2} \geqslant 2\lambda_{\rho-1},$$

and therefore

$$\lambda_\rho - \lambda_{\rho-1} \geqslant \lambda_{\rho-1} - \lambda_{\rho-2}.$$

Assuming that proposition (iii) has been proved for all values of σ up to $(\rho-1)$, it follows that

$$\lambda_\rho - \lambda_{\rho-1} \geqslant \lambda_{\rho-1} - \lambda_{\rho-2} \geqslant \ldots \geqslant \lambda_2 - \lambda_1 \geqslant \lambda_1.$$

Next we take any minor of A of order $(\rho-1)$: let this be

$$L = | a_{jk} | \qquad (j = j_1,\ j_2,\ \ldots\ j_{\rho-1};\ \ k = k_1,\ k_2,\ \ldots\ k_{\rho-1}).$$

From this we can construct, as in VI. 18, a determinant L_{hi} by bordering L with elements of M taken from the row and column in which a_{hi} occurs. Then by Kronecker's theorem (*l.c.*)

$$| L a_{hi} - L_{hi} | = 0 \qquad (h = h_1,\ h_2,\ \ldots\ h_\rho;\ \ i = i_1,\ i_2,\ \ldots\ i_\rho),$$

identically. Expanding in powers of L, we have

$$L^\rho M = L^{\rho-1} M_1 + L^{\rho-2} M_2 + \ldots + M_\rho \quad \ldots\ldots\ldots(1),$$

where M_σ is a homogeneous function in the quantities L_{hi}, with coefficients which are minors of M of order $(\rho-\sigma)$.

Suppose, now, that p^l is the highest power of p contained in L, and $p^{l'}$ the highest power contained in the highest common factor of all the determinants L_{hi}. Then $L^{\rho-\sigma} M_\sigma$ contains p to at least the power τ_σ, where

$$\tau_\sigma = (\rho - \sigma)\, l + l'\sigma + \lambda_{\rho-\sigma} \qquad (\sigma = 1,\ 2,\ \ldots\ \rho),$$

while $L^\rho M$ contains p exactly to the power

$$\tau_0 = \rho l + \lambda_\rho.$$

Hence
$$\begin{aligned}\tau_{\sigma+1} - \tau_\sigma &= (l' - l) - (\lambda_{\rho-\sigma} - \lambda_{\rho-\sigma-1}) \\ &\geqslant (l' - l) - (\lambda_\rho - \lambda_{\rho-1}),\end{aligned}$$

by what has been proved and assumed with regard to M. It follows from this that

$$l' - l \leqslant \lambda_\rho - \lambda_{\rho-1} \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2),$$

because otherwise we should have

$$\tau_\rho > \tau_{\rho-1} > \ldots > \tau_1 > \tau_0,$$

and then every term on the right of (1) would contain p to a higher power than the term on the left, which is impossible.

The relation (2) may also be written

$$\lambda_\rho + l \geqslant \lambda_{\rho-1} + l' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3);$$

now if p^m is the highest power of p contained in all those minors of A of which L is a minor, and which are of order ρ,

$$l' \geqslant m,$$

and consequently $$\lambda_\rho + l \geqslant \lambda_{\rho-1} + m \quad\text{.......................(4).}$$

We will now suppose that L, M are regular with respect to p. Then

$$l = l_{\rho-1}, \quad \lambda_\rho = l_\rho,$$

and the relation (3) becomes

$$l_\rho + l_{\rho-1} \geqslant \lambda_{\rho-1} + l'.$$

Now $$l' \geqslant l_\rho, \quad \lambda_{\rho-1} \geqslant l_{\rho-1},$$

because minors of L, M are also minors of A; so that the three conditions, taken together, lead to

$$l' = l_\rho, \quad \lambda_{\rho-1} = l_{\rho-1} \qquad (\rho = 1,\ 2,\ \ldots\ \rho).$$

We have proved, then, that if $l = l_{\rho-1}$ then $l' = l_\rho$; this is equivalent to proposition (ii). It has also been proved that if $\lambda_\rho = l_\rho$, then $\lambda_{\rho-1} = l_{\rho-1}$; this is equivalent to proposition (i).

Finally, the relation

$$\lambda_\rho - \lambda_{\rho-1} \geqslant \lambda_{\rho-1} - \lambda_{\rho-2},$$

previously established for M, becomes

$$l_\rho - l_{\rho-1} \geqslant l_{\rho-1} - l_{\rho-2},$$

or, with a change of notation,

$$e_\rho \geqslant e_{\rho-1},$$

which is proposition (iii) for $\sigma = \rho$.

Since each element of A is divisible by p^{l_1}, every minor of order 2 is divisible by p^{2l_1}, so that

$$l_2 - l_1 \geqslant l_1,$$

which proves (iii) when $\sigma = 2$. All that has been assumed for the purpose of the induction is the truth of proposition (iii), as applied to M, for $\sigma = 2, 3, \ldots (\rho - 1)$; thus the demonstration is complete.

From the supplementary relation (4) we infer another proposition, namely:—

(iv) If S_ρ and $S_{\rho-1}$ are any two minors of A of orders ρ and $(\rho - 1)$ respectively, then $S_\rho S_{\rho-1}$ is divisible by the product of the

greatest common measure of all minors of S_ρ which are of order $(\rho-1)$ by the greatest common measure of all those minors of A of which $S_{\rho-1}$ is a first minor.

6. It has been supposed hitherto that the value of A is not zero. We shall now remove this restriction.

Unless every element of A is zero there will be at least one complete system of minors of the same order which do not all vanish; while if each minor of order m vanishes, every minor of higher order will also vanish. Thus there will be a definite integer r such that not all the minors of order r vanish, but all the minors of higher order do vanish. This number r is called the *rank* of the determinant A. If all the elements of A are zeros, it is convenient to say that A is of rank 0.

When A is of rank $r\,(<n)$, the modification which has to be made in the foregoing theory is to put

$$D_{r+1}=D_{r+2}=\ldots=D_n=0,$$

$$E_{r+1}=E_{r+2}=\ldots=E_n=0.$$

Every prime factor of D_r will be associated with r finite indices $e_1, e_2, \ldots e_r$ such that

$$e_r \geqslant e_{r-1} \geqslant \ldots \geqslant e_1,$$

and if we put

$$e_1+e_2+\ldots+e_r=l_r,$$

this is the index of the highest power of the prime contained in D_r.

All the propositions of Art. 5 remain true, and the proof of them is still valid; in the enunciations, however, it must be understood that ρ or σ, as the case may be, does not exceed r.

7. The term *elementary factor* (or *elementary divisor*) has not always been used by different writers in the same sense. Sometimes it means one of the quantities E_σ, sometimes one of the powers p^{e_σ}. Frobenius refers to E_σ as the σth elementary factor of A, while by an elementary factor he means one of the powers p^{e_σ}. It may be convenient occasionally to refer to p^{e_σ} as an elementary power-factor, or simply a power-factor of A: but as a rule we shall follow Frobenius's terminology. Of course the

system of the quantities E_σ determines the quantities p^{e_σ} and conversely.

8. Suppose that $A=|a_{nn}|$ and $B=|b_{nn}|$ are any two determinants of the same order, with elements all belonging to the same field. Let their product be expressed, in one of the usual ways, as a determinant $C=|c_{nn}|$ of the same order n. We shall now prove that the σth elementary factor of C is divisible by the σth elementary factor of each of the components A, B.

To fix the ideas, assume that the product AB has been formed by compounding rows of A with columns of B.

Let B_ρ, C_ρ be two corresponding determinant factors of B, C and let β_ρ, γ_ρ be the exponents of the highest powers of the prime p contained in them. Since every minor of C can be expressed as a linear function of minors of B it is clear that $\gamma_\rho \geqslant \beta_\rho$ and consequently C_ρ is divisible by B_ρ. What has to be proved is that

$$\gamma_\rho - \gamma_{\rho-1} \geqslant \beta_\rho - \beta_{\rho-1}.$$

Take L a regular minor of B of order $(\rho-1)$, and M a regular minor of C of order ρ. Let us write

$$L=|b_{jk}| \qquad (j=j_1,\ j_2,\ \ldots\ j_{\rho-1};\quad k=k_1,\ k_2,\ \ldots\ k_{\rho-1}),$$
$$M=|c_{hi}| \qquad (h=h_1,\ h_2,\ \ldots\ h_\rho;\quad i=i_1,\ i_2,\ \ldots\ i_\rho).$$

We proceed to apply to these minors exactly the same argument as that applied in Art. 5 to the determinants there denoted by L, M. Instead of the determinant there denoted by A we shall now use a determinant of order $(2\rho-1)$ which may be briefly written in the form

$$A=\begin{vmatrix} (L) & (P) \\ (Q) & (M) \end{vmatrix},$$

where (P) is a matrix taken from B, with suffixes

$$\begin{matrix} (j_1,\ i_1), & (j_1,\ i_2), & \ldots & (j_1,\ i_\rho) \\ \vdots & \vdots & & \vdots \\ (j_{\rho-1},\ i_1), & (j_{\rho-1},\ i_2), & \ldots & (j_{\rho-1},\ i_\rho), \end{matrix}$$

and (Q) is a matrix taken from C, with suffixes

$$\begin{matrix} (h_1,\ k_1), & (h_1,\ k_2), & \ldots & (h_1,\ k_{\rho-1}) \\ \vdots & \vdots & & \vdots \\ (h_\rho,\ k_1), & (h_\rho,\ k_2), & \ldots & (h_\rho,\ k_{\rho-1}). \end{matrix}$$

Instead of the determinant L_{hi} of Art. 5 we obtain one which consists of the new L bordered in such a way that the last row is

$$c_{h,k_1},\ c_{h,k_2},\ \dots\ c_{h,k_{\rho-1}},\ c_{h,i},$$

while the last column, read from the top, is

$$b_{j_1,i},\ b_{j_2,i},\ \dots\ b_{j_{\rho-1},i},\ c_{h,i}.$$

If, now, $p^{l'}$ is the highest power of p contained in all these last determinants, the formula (3) of Art. 5, applied to the present case, gives

$$\gamma_\rho + \beta_{\rho-1} \geqslant \gamma_{\rho-1} + l'.$$

But each of the determinants corresponding to L_{hi} can be expressed as a linear homogeneous function of minors of B of the order ρ; hence $l' \geqslant \beta_\rho$ and consequently

$$\gamma_\rho + \beta_{\rho-1} \geqslant \gamma_{\rho-1} + \beta_\rho,$$

or

$$\gamma_\rho - \gamma_{\rho-1} \geqslant \beta_\rho - \beta_{\rho-1}.$$

This proves that every elementary factor of C is divisible by the corresponding elementary factor of B. By similar reasoning it may be shewn that each factor of A is contained in the corresponding factor of C.

The theorem may be obviously extended to the case when C is compounded of three or more determinants of the same order and belonging to the same field.

9. A very important special case is when the value of one of the determinants A, B is unity. Suppose that $A = 1$; then the elementary factors of C are identical with those of B.

To see this, let E_σ and E_σ' be any two corresponding elementary factors of B, C respectively. Then, by the theorem of Art. 5, E_σ' is divisible by E_σ. But since (A) is a unit matrix, we can find another matrix $(\bar{A})$, with integral elements, such that

$$(\bar{A})\ (A) = [1],$$

and hence

$$(\bar{A})\ (C) = (\bar{A})\ (A)\ (B) = (B),$$

identically. Therefore E_σ is divisible by E_σ', and finally $E_\sigma' = E_\sigma$.

Two determinants of which one can be identically transformed into the other by compounding it with a unit determinant are

often said to be *equivalent.* Thus equivalent determinants have the same elementary factors, and therefore also the same determinant factors.

10. It will now be supposed that the elements of the determinant A are ordinary whole numbers. We shall prove that it is possible to find two unitary matrices (P), (Q), such that

$$(P)(A)(Q) = (E) = [E_n],$$

where (E) or $[E_n]$ denotes a matrix of which the principal diagonal consists of the elements $E_1, E_2, \dots E_n$ (the elementary divisors of A) while all its other elements are zeros.

Consider the effect of transforming any determinant by one of the four following elementary operations:—

(1) Interchanging two rows;

(2) Interchanging two columns;

(3) Adding to any row k times another row, where k is any integer, positive or negative;

(4) Adding to any column k times another column.

Any one of these transformations leaves unaltered the values of the elementary factors E_σ, because the complete system of minors of any order for the new determinant coincides, except as to arrangement, with the same system for the original determinant. Moreover, by the rule for the composition of matrices (v. 2) it is easily proved that if D' is the determinant derived from D by either of the operations (1), (3) there is an identity

$$(D') = (P)(D)$$

with $|P| = +1$ or -1. Similarly, if D'' is derived from D by one of the operations (2), (4), there is an identity

$$(D'') = (D)(Q)$$

with $|Q| = +1$ or -1.

In the given determinant $|a_{nn}|$, unless every element is zero, there will be at least one element of which the numerical value is equalled or surpassed by every other element which does not vanish. By the application of one or both of the operations (1), (2) this element can be brought to the head of the principal diagonal.

By adding multiples of the first row to the other rows, we can reduce every element in the first column, except the one at the top, to a value which is either zero, or numerically less than the value of the element at the top. We now interchange rows until the first column is headed by an element, not zero, of the smallest numerical value; and then proceed, as before, to reduce the other elements of the first column. This process must eventually come to an end; so that, after a finite number of transformations, we get a first column in which every element is zero except the one at the top.

We now operate upon the transformed determinant so as to reduce the values of the elements in the first row by means of the elementary operations (2) and (4). Ultimately we get a first row of the form

$$a,\ 0,\ 0, \ldots 0$$

and a is certainly not greater, numerically, than the number previously in the same place.

If the elements now below a in the first column are not all zero, we reduce them, as before, by the operations (1) and (3), until we have a column with a' at the top, and all the other elements zero.

The elements a, a', etc., which we thus get at the head of the leading diagonals, diminish continually in numerical value, so that a stage must come when no further reduction is possible. When this is so, every element in the first row is zero except the first and at the same time every element in the first column is zero except the first. Thus A has been transformed into A', where

$$A' = \begin{vmatrix} \eta, & 0, & 0, & \ldots & 0 \\ 0, & b_{22}, & b_{23}, & \ldots & b_{2n} \\ 0, & b_{32}, & b_{33}, & \ldots & b_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & b_{n2}, & b_{n3}, & \ldots & b_{nn} \end{vmatrix},$$

and η is different from zero.

Suppose, now, that there is an element b_{ik} which is not a multiple of η: then we add the $(k+1)$th row of A' to the first row,

and carry out the process of reduction, as before, until we get another determinant like A': this will have, instead of η, an element of less numerical value. Here again the process must come to a stop after a finite number of operations: so that we ultimately get a determinant A' where every element b_{ik} is a multiple of η.

If we now put $\eta = e_1$, $b_{ik} = e_1 c_{ik}$, then

$$A' = \begin{vmatrix} e_1, & 0, & 0, & \dots & 0 \\ 0, & e_1 c_{22}, & e_1 c_{23}, & \dots & e_1 c_{2n} \\ 0, & e_1 c_{32}, & e_1 c_{33}, & \dots & e_1 c_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0, & e_1 c_{n2}, & e_1 c_{n3}, & \dots & e_1 c_{nn} \end{vmatrix},$$

and if we put

$$C = | c_{ik} | \qquad (i,\ k = 2,\ 3,\ \dots\ n),$$

the determinant C can be treated in exactly the same way, so as to transform it into

$$C' = \begin{vmatrix} e_2, & 0, & 0, & \dots & 0 \\ 0, & e_2 d_{33}, & e_2 d_{34}, & \dots & e_2 d_{3n} \\ 0, & e_2 d_{43}, & e_2 d_{44}, & \dots & e_2 d_{4n} \\ \dots & \dots & \dots & \dots & \dots \\ 0, & e_2 d_{n3}, & e_2 d_{n4}, & \dots & e_2 d_{nn} \end{vmatrix},$$

and there will be a corresponding change of A' into

$$\begin{vmatrix} e_1, & 0, & 0, & \dots & 0 \\ 0, & e_1 e_2, & 0, & \dots & 0 \\ 0, & 0, & \alpha_{33}, & \dots & \alpha_{3n} \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & \alpha_{n3}, & \dots & \alpha_{nn} \end{vmatrix},$$

where each element α_{ik} is a multiple of $e_1 e_2$.

By continuing the argument we arrive at the final result that (A) can be transformed into the equivalent diagonal system

$$[E_1,\ E_2,\ \dots\ E_n],$$

where
$$E_\sigma = e_1 e_2 \dots e_\sigma,$$

and $e_1, e_2, \dots e_n$ are integers.

It has been assumed that some one element of A is not zero

this being so, e_1 is certainly not zero. But the determinant above denoted by C may have every element zero: in this case we put

$$e_2 = e_3 = \dots = e_n = 0.$$

In like manner, after getting finite integers $e_1, e_2, \dots e_r$ the determinant which at that stage corresponds to C may have all its elements zero: we then put

$$e_{r+1} = e_{r+2} = \dots = e_n = 0.$$

11. The reduction of (A) to the normal form (E) having been effected by the four elementary operations, we have an identity

$$(E) = (P_h)(P_{h-1}) \dots (P_1)(A)(Q_1)(Q_2) \dots (Q_k),$$

and since the composition of matrices is associative this may be put into the form

$$(E) = (P)(A)(Q),$$

where (P), (Q) are unit matrices.

Hence the σth elementary divisor of (A) coincides with the σth elementary divisor of (E). But it is easy to see that the latter is E_σ. For, in the first place, the non-vanishing minors of (E) of order σ are simply the products of E_1, E_2, etc. taken σ at a time; and of these the greatest common measure is

$$D_\sigma = E_1 E_2 \dots E_\sigma.$$

Hence the σth elementary divisor is $D_\sigma / D_{\sigma-1}$, that is, E_σ.

It follows that two systems (A), (B), of the type here considered, which have the same set of elementary factors must be equivalent. For if (E) is the diagonal system of elementary divisors there will be four unit matrices (P), (Q), (R), (S) such that

$$(E) = (P)(A)(Q) = (R)(B)(S),$$

and hence $$(B) = (R)^{-1}(P)(A)(Q)(S)^{-1},$$

where $(R)^{-1}$ is the unit matrix such that

$$(R)^{-1}(R) = [1],$$

and $(S)^{-1}$ is similarly defined.

12. The argument and results of Arts. 10, 11 apply, *mutatis mutandis*, to the case when the elements of the determinants

considered are not integers, but rational integral functions of one or more variables. In fact the process of reduction to the normal form (E) can always be carried out when the greatest common measure of any two of the elements can be found by a process of chain-division analogous to that which is used for two ordinary integers.

13. The elementary divisors of a determinant may also be regarded as being associated with its matrix; and the theory may be extended to rectangular matrices in general. All that is necessary is to enlarge the matrices by adding rows or columns of zero elements until they become square.

CHAPTER VIII.

DETERMINANTS OF SPECIAL FORMS.

1. WHEN a square array is written down, it is natural to inquire what simplifications arise in the determinant of the array when special relations are supposed to exist between the elements. And looking at the figure the relations which naturally suggest themselves are those which depend on the geometrical form which the array assumes. Hence we have various forms of determinants obtained by supposing relationships, of equality or otherwise, to exist between elements situated symmetrically in the figure; this shews how the notation employed has influenced the development of the theory.

The most important of these special forms are symmetrical and skew symmetrical determinants. Here the special form of geometrical symmetry considered is with regard to the leading diagonal. Elements which are situated in regard to the diagonal in the position of a point and its image with respect to a mirror coinciding with the diagonal, have been called conjugate: two such elements are denoted by a_{ik} and a_{ki}.

2. If $a_{ik} = a_{ki}$, the determinant is called symmetrical.

The square of any determinant may be expressed as a symmetrical determinant of the same order.

For $$|a_{ik}|^2 = |c_{ik}|$$

where $$c_{ik} = a_{i1}a_{k1} + a_{i2}a_{k2} + \dots$$
$$= c_{ki}.$$

It follows from this that every even power of a determinant is a symmetrical determinant.

3. We may also suppose the determinant to be symmetrical with respect to the centre of the square formed by the elements of the determinant.

Two cases arise, according as the determinant is of even or of odd order.

First, if the order of the determinant is $2r$, we may write it in the form:

$$D = \begin{vmatrix} a_1, & b_1, & c_1 \ldots m_1, & n_1, & \nu_1, & \mu_1 \ldots \gamma_1, & \beta_1, & \alpha_1 \\ a_2, & b_2, & c_2 \ldots m_2, & n_2, & \nu_2, & \mu_2 \ldots \gamma_2, & \beta_2, & \alpha_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_r, & b_r, & c_r \ldots m_r, & n_r, & \nu_r, & \mu_r \ldots \gamma_r, & \beta_r, & \alpha_r \\ \alpha_r, & \beta_r, & \gamma_r \ldots \mu_r, & \nu_r, & n_r, & m_r \ldots c_r, & b_r, & a_r \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_2, & \beta_2, & \gamma_2 \ldots \mu_2, & \nu_2, & n_2, & m_2 \ldots c_2, & b_2, & a_2 \\ \alpha_1, & \beta_1, & \gamma_1 \ldots \mu_1, & \nu_1, & n_1, & m_1 \ldots c_1, & b_1, & a_1 \end{vmatrix}.$$

In this determinant add the last column to the first, the last but one to the second, the $(r+1)$st to the rth, then it becomes

$$D = \begin{vmatrix} a_1+\alpha_1, & b_1+\beta_1 \ldots n_1+\nu_1, & \nu_1, & \mu_1 \ldots \beta_1, & \alpha_1 \\ a_2+\alpha_2, & b_2+\beta_2 \ldots n_2+\nu_2, & \nu_2, & \mu_2 \ldots \beta_2, & \alpha_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_r+\alpha_r, & b_r+\beta_r \ldots n_r+\nu_r, & \nu_r, & \mu_r \ldots \beta_r, & \alpha_r \\ a_r+\alpha_r, & b_r+\beta_r \ldots n_r+\nu_r, & n_r, & m_r \ldots b_r, & a_r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_2+\alpha_2, & b_2+\beta_2 \ldots n_2+\nu_2, & n_2, & m_2 \ldots b_2, & a_2 \\ a_1+\alpha_1, & b_1+\beta_1 \ldots n_1+\nu_1, & n_1, & m_1 \ldots b_1, & a_1 \end{vmatrix}.$$

Now subtract the first row from the last, the second from the last but one, the rth from the $(r+1)$st, then

$$D = \begin{vmatrix} a_1+\alpha_1, & b_1+\beta_1 \ldots n_1+\nu_1, & \nu_1, & \mu_1 \ldots \beta_1, & \alpha_1 \\ a_2+\alpha_2, & b_2+\beta_2 \ldots n_2+\nu_2, & \nu_2, & \mu_2 \ldots \beta_2, & \alpha_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_r+\alpha_r, & b_r+\beta_r \ldots n_r+\nu_r, & \nu_r, & \mu_r \ldots \beta_r, & \alpha_r \\ 0, & 0 \ldots 0, & n_r-\nu_r, & m_r-\mu_r \ldots b_r-\beta_r, & a_r-\alpha_r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0 \ldots 0, & n_2-\nu_2, & m_2-\mu_2 \ldots b_2-\beta_2, & a_2-\alpha_2 \\ 0, & 0 \ldots 0, & n_1-\nu_1, & m_1-\mu_1 \ldots b_1-\beta_1, & a_1-\alpha_1 \end{vmatrix}.$$

Hence (IV. 7),

$$D = \begin{vmatrix} a_1 + \alpha_1 \dots n_1 + \nu_1 \\ \dots\dots\dots\dots \\ a_r + \alpha_r \dots n_r + \nu_r \end{vmatrix} \cdot \begin{vmatrix} n_r - \nu_r \dots a_r - \alpha_r \\ \dots\dots\dots\dots \\ n_1 - \nu_1 \dots a_1 - \alpha_1 \end{vmatrix}.$$

But if the order of the determinant is $2r+1$, it may be written in the form

$$D = \begin{vmatrix} a_1, & b_1 \dots n_1, & u_1, & \nu_1 \dots \beta_1, & \alpha_1 \\ a_2, & b_2 \dots n_2, & u_2, & \nu_2 \dots \beta_2, & \alpha_2 \\ \dots & \dots & \dots & \dots & \dots \\ a_r, & b_r \dots n_r, & u_r, & \nu_r \dots \beta_r, & \alpha_r \\ v_1, & v_2 \dots v_r, & p, & v_r \dots v_2, & v_1 \\ \alpha_r, & \beta_r \dots \nu_r, & u_r, & n_r \dots b_r, & a_r \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_1, & \beta_1 \dots \nu_1, & u_1, & n_1 \dots b_1, & a_1 \end{vmatrix}.$$

By proceeding exactly as in the former case, we can shew that

$$D = \begin{vmatrix} a_1 + \alpha_1 \dots n_1 + \nu_1, & u_1 \\ \dots\dots\dots\dots & \\ a_r + \alpha_r \dots n_r + \nu_r, & u_r \\ 2v_1, \dots 2v_r, & p \end{vmatrix} \cdot \begin{vmatrix} n_r - \nu_r \dots a_r - \alpha_r \\ \dots\dots\dots\dots \\ n_1 - \nu_1 \dots a_1 - \alpha_1 \end{vmatrix}.$$

So that when a determinant is symmetrical with respect to the centre of the square formed by its elements, it reduces to the product of two other determinants.

4. If in a determinant the conjugate elements are equal in magnitude but opposite in sign, i.e. if

$$a_{ik} = -a_{ki},$$

the determinant is called a skew determinant. If, moreover,

$$a_{ii} = 0,$$

the determinant is called a skew symmetrical determinant.

5. It will be useful to notice the connexion between two minors of these systems, such that the rows and columns suppressed to obtain the one minor correspond to the columns and rows suppressed to obtain the other. Two such minors may be denoted by

$$P = \begin{vmatrix} a_{pf}, & a_{pg} \dots \\ a_{qf}, & a_{qg} \dots \\ \dots & \dots \end{vmatrix}, \quad Q = \begin{vmatrix} a_{fp}, & a_{fq} \dots \\ a_{gp}, & a_{gq} \dots \\ \dots & \dots \end{vmatrix}.$$

6. If the determinant is symmetrical,

i.e. if $$a_{ik} = a_{ki},$$

clearly $$P = Q.$$

A special case of this is, that in a symmetrical determinant

$$A_{ik} = A_{ki},$$

for A_{ik} is got by suppressing the ith row and kth column, while A_{ki} is got by suppressing the kth row and ith column, thus these determinants are of the same nature as P and Q, and are therefore equal. Thus the determinant of the reciprocal system is also symmetrical. If A is the determinant of the system,

$$\begin{aligned}\frac{dA}{da_{ik}} &= A_{ik} + A_{ki}\frac{da_{ki}}{da_{ik}}\\ &= 2A_{ik}.\end{aligned}$$

But $$\frac{dA}{da_{ii}} = A_{ii}.$$

In a symmetrical determinant A_{ii} and the like are still symmetrical determinants.

7. If in Art. 5 $$a_{ik} = -a_{ki},$$

we see that

$$P = \begin{vmatrix} a_{pf}, & a_{pg} & \cdots \\ a_{qf}, & a_{qg} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = \begin{vmatrix} -a_{fp}, & -a_{gp} & \cdots \\ -a_{fq}, & -a_{gq} & \cdots \\ \cdots & \cdots & \cdots \end{vmatrix} = (-1)^m Q,$$

m being the order of the minors. Thus if m is even

$$P = Q,$$

but if m is odd $$P = -Q.$$

8. The calculation of skew determinants reduces to that of skew symmetrical determinants, which we shall therefore now consider. A skew symmetrical determinant of odd order vanishes, for if we multiply each row by -1, since $a_{ik} = -a_{ki}$, this changes the rows into columns, which does not alter the value of the determinant.

Hence, if n be its order,

$$A = (-1)^n A\,;$$

and therefore $A = 0$ if n is odd.

The minor A_{ik} differs from A_{ki} by the sign of every element; hence

$$A_{ik} = (-1)^{n-1} A_{ki}.$$

Thus $A_{ki} = A_{ik}$ if n is odd, but $= -A_{ik}$ if n is even.

Thus the reciprocal system is skew if n is even, but symmetrical if n is odd.

A_{ii} is a skew symmetrical determinant of order $n-1$, and hence vanishes if n is even.

We have

$$\begin{aligned}\frac{dA}{da_{ik}} &= A_{ik} + A_{ki}\frac{da_{ki}}{da_{ik}}\\ &= A_{ik} - A_{ki}\\ &= 2A_{ik} \text{ if } n \text{ is even}\\ &= 0 \text{ if } n \text{ is odd.}\end{aligned}$$

9. A skew symmetrical determinant of even order is a complete square.

For if $$A = |\, a_{ik} \,|$$
is the determinant, A_{11} vanishes because it is a skew symmetrical determinant of odd order. Hence (VI. 6), if α_{ik} is the complement of a_{ik} in A_{11},

$$\begin{vmatrix} \alpha_{ii}, & \alpha_{ik} \\ \alpha_{ki}, & \alpha_{kk} \end{vmatrix} = 0, \quad \text{or} \quad \alpha_{ii}\alpha_{kk} = \alpha_{ik}^2,$$

since $\alpha_{ik} = \alpha_{ki}$ (Art. 8).

Now by (IV. 24) if we expand according to products of elements in the first row and first column, since $A_{11} = 0$

$$A = -\Sigma a_{1i} a_{k1} \alpha_{ik},$$

where i, k take the values $2, 3 \dots n$;

or $$\begin{aligned}A &= \Sigma a_{1i} a_{1k} \sqrt{\alpha_{ii}\alpha_{kk}}\\ &= \{\Sigma a_{1i} \sqrt{\alpha_{ii}}\}^2.\end{aligned}$$

Thus A is the square of a linear function of the elements of a row. Now α_{ii} is a determinant of order $n-2$, which is even if n is even. Thus a skew symmetrical determinant of order n will

be the square of a rational function of its elements if one of order $n-2$ is so. But when $n=2$,

$$\begin{vmatrix} 0, & a_{12} \\ a_{21}, & 0 \end{vmatrix} = a_{12}^2.$$

Thus skew symmetrical determinants of orders 4, 6 ... $2r$ are squares of rational functions of their elements.

10. Since if $n=2$ the square root contains one term, when $n=4$ the square root will contain 3, when $n=6$ it will contain 5 . 3 terms, and so on. Hence a skew symmetrical determinant of even order n is the square of an aggregate of

$$1 . 3 . 5 \dots (n-1)$$

terms, each consisting of the product of $\frac{1}{2}n$ elements of A.

In particular $a_{12}a_{34} \dots a_{n-1n}$ is a term of $\sqrt{A}$, for

$$(a_{12}a_{34} \dots a_{n-1n})^2 = (-1)^{\frac{n}{2}} a_{12}a_{34} \dots a_{n-1n}a_{21}a_{43} \dots a_{nn-1}.$$

11. This function $\sqrt{A}$ is of importance in analysis, and has been called a Pfaffian by Prof. Cayley on account of the use made of it by Jacobi in his discussion of Pfaff's problem.

That value of $\sqrt{A}$ which contains $a_{12}a_{34} \dots a_{n-1n}$ as first term with positive sign will be denoted by

$$P = [1,\ 2 \dots n].$$

The remaining terms of P are got from the first term,

$$a_{12}a_{34} \dots a_{n-1n},$$

by interchanging all the suffixes 2, 3 ... n in all possible ways, and giving a sign corresponding to the number of inversions. Since $a_{ik} = -a_{ki}$ it is possible to effect the interchange in such a way that all the terms are positive.

The Pfaffian changes sign on interchanging only two suffixes i and k. For if we interchange i and k in the determinant, this interchanges the ith and kth rows as well as the ith and kth columns, thus the value of the determinant remains unchanged. If P_1 is the new value of P,

$$P_1^2 = P^2.$$

Hence $$P_1 = \pm P.$$

To determine which sign we are to take, let us consider the aggregate of terms $a_{ik}p_{ik}$ which contain a_{ik}. Then p_{ik} only contains terms whose suffixes are independent of i and k. The corresponding aggregate for P_1 is

$$a_{ki}p_{ik},$$

which, in consequence of the relation $a_{ki} = -a_{ik}$, proves that

$$P_1 = -P.$$

12. The minor α_{ii} is also a skew symmetrical determinant. We shall shew that

$$\sqrt{\alpha_{ii}} = (-1)^i [2, \ldots i-1, \; i+1, \ldots n],$$

or with $i-2$ cyclical interchanges

$$\sqrt{\alpha_{ii}} = [i+1, \ldots n, \; 2 \ldots i-1].$$

Since $$\alpha_{ik}^2 = \alpha_{ii}\alpha_{kk},$$

it follows that the terms of the product $\sqrt{\alpha_{ii}}\sqrt{\alpha_{kk}}$ are either equal to those of α_{ik}, or equal with opposite signs.

Now the product

$$(-1)^{i+k}[2 \ldots i-1, \; i+1 \ldots n][2 \ldots k-1, \; k+1 \ldots n]$$

and the determinant

$$\alpha_{ik} = \begin{vmatrix} a_{2,2} \ldots\ldots a_{2,k-1}, & a_{2,k+1} \ldots\ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ a_{i-1,2} \ldots a_{i-1,k-1}, & a_{i-1,k+1} \ldots \\ a_{i+1,2} \ldots a_{i+1,k-1}, & a_{i+1,k+1} \ldots \\ \ldots\ldots\ldots\ldots & \ldots\ldots\ldots \end{vmatrix} (-1)^{i+k},$$

by the same number of interchanges of two suffixes, become respectively

$$[k, \; p \,.\, q, \; r, \; s \ldots u, \; v][p, \; q, \; r, \; s \ldots v, \; i]$$

and

$$\begin{vmatrix} a_{kp}, & a_{kq}, & a_{kr} \ldots a_{ki} \\ a_{pp}, & a_{pq}, & a_{pr} \ldots a_{pi} \\ \ldots & \ldots & \ldots \\ a_{vp}, & a_{vq}, & a_{vr} \ldots a_{vi} \end{vmatrix}.$$

And the term

$$a_{kp}a_{qr} \ldots a_{uv} \,.\, a_{pq}a_{rs} \ldots a_{vi}$$

of the product agrees in sign with the first term of the determinant

$$a_{kp}a_{pq}a_{qr} \ldots a_{vi},$$

whence the theorem follows.

This proposition serves to determine $\sqrt{\alpha_{11}}$, $\sqrt{\alpha_{22}}$ as functions free from ambiguity of sign.

13. Since we have shewn in Art. 9 that

$$\sqrt{A} = a_{12}\sqrt{\alpha_{22}} + a_{13}\sqrt{\alpha_{33}} + \dots + a_{1n}\sqrt{\alpha_{nn}},$$

it follows that

$$[1,\ 2 \dots n] = a_{12}[3 \dots n] + a_{13}[4 \dots n,\ 2] + \dots + a_{1n}[2 \dots n-1];$$

a relation which enables us to determine Pfaffians of order n from those of order $n-2$.

Observe that after we have selected the suffix 1, the others are written cyclically. Hence

$$\begin{aligned}
[1,\ 2] &= a_{12} \\
[1,\ 2,\ 3,\ 4] &= a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23} \\
[1,\ 2,\ 3,\ 4,\ 5,\ 6] &= a_{12}[3,\ 4,\ 5,\ 6] + a_{13}[4,\ 5,\ 6,\ 2] + a_{14}[5,\ 6,\ 2,\ 3] \\
&\qquad + a_{15}[6,\ 2,\ 3,\ 4] + a_{16}[2,\ 3,\ 4,\ 5] \\
&= a_{12}a_{34}a_{56} + a_{12}a_{35}a_{64} + a_{12}a_{36}a_{45} \\
&\quad + a_{13}a_{45}a_{62} + a_{13}a_{46}a_{25} + a_{13}a_{42}a_{56} \\
&\quad + a_{14}a_{56}a_{23} + a_{14}a_{52}a_{36} + a_{14}a_{53}a_{62} \\
&\quad + a_{15}a_{62}a_{34} + a_{15}a_{63}a_{42} + a_{15}a_{64}a_{23} \\
&\quad + a_{16}a_{23}a_{45} + a_{16}a_{24}a_{53} + a_{16}a_{25}a_{34}.
\end{aligned}$$

In particular

$$\begin{vmatrix} 0, & a, & -b, & c \\ -a, & 0, & f, & e \\ b, & -f, & 0, & d \\ -c, & -e, & -d, & 0 \end{vmatrix} = (ad + be + cf)^2.$$

14. In a skew symmetrical determinant of even order, A_{ii} vanishes, being a skew symmetrical determinant of odd order.

But (Art. 8),

$$\begin{aligned}
A_{ik} &= \tfrac{1}{2}\frac{dA}{da_{ik}} \\
&= \tfrac{1}{2}\frac{d}{da_{ik}}[1,\ 2 \dots n]^2 \\
&= [1,\ 2 \dots n]\frac{d}{da_{ik}}[1,\ 2 \dots n].
\end{aligned}$$

Now

$$\begin{aligned}
P = [1,\ 2 \dots n] &= (-1)^{i-1}[i,\ 1 \dots i-1,\ i+1 \dots n] \\
&= (-1)^{i-1}\{a_{i1}[2 \dots i-1,\ i+1 \dots n] + \dots \\
&\quad + a_{ik}(-1)^{k-1}[1,\ 2 \dots i-1,\ i+1 \dots k-1,\ k+1 \dots n] + \dots\};
\end{aligned}$$

hence $$A_{ik} = (-1)^{i+k} [1,\ 2 \dots n] \{ik\},$$
where $\{ik\}$ is the Pfaffian got by omitting i and k in $[1, 2 \dots n]$.

15. In a skew symmetrical determinant of odd order A_{ii} is a skew symmetrical determinant of even order, and is hence the square of a Pfaffian;

viz. $$A_{ii} = [1 \dots i-1,\ i+1 \dots n]^2,$$
$$\sqrt{A_{ii}} = (-1)^{i-1} [1 \dots i-1,\ i+1 \dots n]$$
$$= [i+1 \dots n,\ 1 \dots i-1].$$

Also, since $$A = 0,$$
$$A_{ik}^2 = A_{ii} A_{kk}.$$

Hence $$A_{ik} = [i+1 \dots n,\ 1 \dots i-1] [k+1 \dots n,\ 1 \dots k-1].$$

16. The result of bordering a skew symmetrical determinant is also of interest. The result assumes different forms according as the determinant which we border is of odd or even order.

Let the original skew symmetrical determinant be
$$A = |\, a_{ik} \,|,$$
and let the bordered determinant be
$$D = \begin{vmatrix} a_{\alpha\beta}, & a_{\alpha 1}, & a_{\alpha 2}, & a_{\alpha 3} \dots \\ a_{1\beta}, & a_{11}, & a_{12}, & a_{13} \dots \\ a_{2\beta}, & a_{21}, & a_{22}, & a_{23} \dots \\ a_{3\beta}, & a_{31}, & a_{32}, & a_{33} \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

By Cauchy's theorem (III. 24)
$$D = a_{\alpha\beta} A - \Sigma a_{\alpha i}\, a_{k\beta} A_{ik}.$$

Now, if A is of odd order it vanishes, and
$$A_{ik} = [i+1 \dots n,\ 1 \dots i-1] [k+1 \dots n,\ 1 \dots k-1];$$
hence, if we suppose that $a_{\beta k} = -a_{k\beta}$,
$$D = \Sigma a_{\alpha i} a_{\beta k} [i+1 \dots n,\ 1 \dots i-1] [k+1 \dots n,\ 1 \dots k-1]$$
$$= (a_{\alpha 1} [2,\ 3 \dots n] + \dots)(a_{\beta 1} [2,\ 3 \dots n] + \dots)$$
$$= [\alpha,\ 1,\ 2 \dots n] [\beta,\ 1,\ 2 \dots n],$$

where in the Pfaffians such expressions as $a_{i\alpha}$, $a_{\beta k}$ which do not occur in the determinant are supposed to mean $-a_{\alpha i}$, $-a_{k\beta}$.

But if A is of even order,

$$D = a_{\alpha\beta}[1, 2 \ldots n]^2 + \Sigma a_{\alpha i} a_{\beta k}(-1)^{i+k}\{ik\}[1, 2 \ldots n] \quad \text{(Art. 14)}$$
$$= [1, 2 \ldots n][\alpha, \beta, 1, 2 \ldots n].$$

17. We have hitherto treated of skew symmetrical determinants: it is easy to reduce to these the calculation of skew determinants. Namely, by IV. 23

$$D' = D + \Sigma a_{ii} D_i + \Sigma a_{ii} a_{kk} D_{ik} + \ldots + a_{11} a_{22} \ldots a_{nn},$$

where D is what D' becomes when all the diagonal elements vanish. D_i is what the coefficient of a_{ii} in D' becomes when the diagonal elements vanish; D_{ik} the coefficient of $a_{ii}a_{kk}$ in D' with the elements in the leading diagonal zeros, and so on.

If all the elements in the leading diagonal are equal to x we can write this

$$D' = x^n + x^{n-2}\Sigma D_2 + x^{n-4}\Sigma D_4 + \ldots + x^{n-m}\Sigma D_m + \ldots$$

where D_m is a minor of order m got by suppressing $n-m$ rows and columns which meet in a diagonal element, the other diagonal elements being put zero, and the summation extends to all m-ads in n.

If m is odd, D_m vanishes, and if m is even it is a complete square.

Thus, the elements being skew,

$$\begin{vmatrix} x, & a_{12}, & a_{13} \\ a_{21}, & x, & a_{23} \\ a_{31}, & a_{32}, & x \end{vmatrix} = x^3 + x(a^2_{12} + a^2_{13} + a^2_{23})$$

$$\begin{vmatrix} x, & a_{12}, & a_{13}, & a_{14} \\ a_{21}, & x, & a_{23}, & a_{24} \\ a_{31}, & a_{32}, & x, & a_{34} \\ a_{41}, & a_{42}, & a_{43}, & x \end{vmatrix} = x^4 + x^2(a^2_{12} + a^2_{13} + a^2_{14} + a^2_{23} + a^2_{24} + a^2_{34}) \\ + (a_{12}a_{34} + a_{13}a_{42} + a_{14}a_{23})^2.$$

18. We can apply this last theorem to prove Euler's theorem concerning the product of two numbers, each of which is the sum of four squares. Namely, we have

$$\begin{vmatrix} a, & b, & c, & d \\ -b, & a, & -d, & c \\ -c, & d, & a, & -b \\ -d, & -c, & b, & a \end{vmatrix} = (a^2+b^2+c^2+d^2)^2,$$

$$\begin{vmatrix} p, & q, & r, & s \\ -q, & p, & -s, & r \\ -r, & s, & p, & -q \\ -s, & -r, & q, & p \end{vmatrix} = (p^2+q^2+r^2+s^2)^2.$$

Now multiply these two determinants by rows, then if we write

$$A = ap + bq + cr + ds, \qquad B = -aq + bp - cs + dr,$$

$$C = -ar + bs + cp - dq, \quad D = -as - br + cq + dp,$$

we get a skew determinant of the same form as the other two, whose value is

$$(A^2 + B^2 + C^2 + D^2)^2,$$

whence

$$(a^2+b^2+c^2+d^2)(p^2+q^2+r^2+s^2) = A^2 + B^2 + C^2 + D^2.$$

If we were to effect the multiplication by rows and columns we should get another form of the same theorem; by suitably permuting the rows and columns we get still further representations of the way in which the product of two numbers, each of which is the sum of four squares, can be represented as the sum of four squares.

19. We have seen that the square of any determinant is a symmetrical determinant (Art. 2). Cayley and Brioschi have shewn independently that the square of a determinant of even order can be represented by a skew symmetrical determinant of even order.

Brioschi's method is as follows: We have

$$A = \begin{vmatrix} a_{11}, & a_{12} \dots a_{1n-1}, & a_{1n} \\ a_{21}, & a_{22} \dots a_{2n-1}, & a_{2n} \\ \dots & \dots & \dots \\ a_{n1}, & a_{n2} \dots a_{nn-1}, & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{12}, & -a_{11} \dots a_{1n}, & -a_{1n-1} \\ a_{22}, & -a_{21} \dots a_{2n}, & -a_{2n-1} \\ \dots & \dots & \dots \\ a_{n2}, & -a_{n1} \dots a_{nn}, & -a_{nn-1} \end{vmatrix}.$$

Multiply these two equal determinants together by rows, and we obtain:

$$A^2 = \begin{vmatrix} 0, & l_{12}, & l_{13} \dots l_{1n} \\ l_{21}, & 0, & l_{23} \dots l_{2n} \\ \dots & \dots & \dots \\ l_{n1}, & l_{n2}, & l_{n3} \dots 0 \end{vmatrix},$$

where

$$l_{rs} = a_{r1}a_{s2} - a_{r2}a_{s1} + a_{r3}a_{s4} - a_{r4}a_{s3} + \dots + a_{rn-1}a_{sn} - a_{rn}a_{sn-1},$$

so that $$l_{ss} = 0, \quad l_{rs} + l_{sr} = 0.$$

Thus A^2 is represented as a skew symmetrical determinant. It follows that A can be represented as a Pfaffian of the functions l. If $n = 4$, for example,

$$\begin{vmatrix} a_{11} & \dots & a_{14} \\ \dots & \dots & \dots \\ a_{41} & \dots & a_{44} \end{vmatrix} = l_{12}l_{34} + l_{13}l_{42} + l_{14}l_{23}.$$

The sign is determined by making the sign of a single term in the determinant and Pfaffian agree.

If, instead of interchanging columns, we interchanged rows, we should get another independent representation of the determinant as a Pfaffian.

20. A third class of determinants comprisés those of the form

$$D = \begin{vmatrix} a_1, & a_2, & a_3 & \dots a_n \\ a_2, & a_3, & a_4 & \dots a_{n+1} \\ a_3, & a_4, & a_5 & \dots a_{n+2} \\ \dots & \dots & \dots & \dots \\ a_n, & a_{n+1}, & a_{n+2} & \dots a_{2n-1} \end{vmatrix},$$

where all the elements in a line at right angles to the leading diagonal are the same. If the elements had been written with double suffixes we should have had the relation

$$a_{pq} = a_{rs}$$

whenever $$p + q = r + s.$$

Such determinants have been called orthosymmetrical. Their most important property is that we can replace the elements by differences of a_1.

For if we operate on the rows as we did in Chap. III. 5 (iv), and put

$$\Delta a_x = a_{x+1} - a_x, \text{ \&c.}$$

$$D = \begin{vmatrix} a_1, & a_2, & \dots & a_n \\ \Delta a_1, & \Delta a_2, & \dots & \Delta a_n \\ \Delta^2 a_1, & \Delta^2 a_2, & \dots & \Delta^2 a_n \\ \dots & \dots & \dots & \dots \\ \Delta^{n-1} a_1, & \Delta^{n-1} a_2, & \dots & \Delta^{n-1} a_n \end{vmatrix}.$$

Now repeat the same series of operations on the columns, beginning at the last, then

$$D = \begin{vmatrix} a_1, & \Delta a_1, & \dots & \Delta^{n-1} a_1 \\ \Delta a_1, & \Delta^2 a_1, & \dots & \Delta^n a_1 \\ \Delta^2 a_1, & \Delta^3 a_1, & \dots & \\ \dots & \dots & \dots & \\ \Delta^{n-1} a_1, & \Delta^n a_1, & \dots & \Delta^{2n-2} a_1 \end{vmatrix}.$$

An important example of this class of determinants is that where a_k is a function of k of the mth degree in k, whose highest term has coefficient unity, so that the quantities $a_1, a_2 \dots$ form an arithmetic series of the mth order. If $m = n - 1$ all the elements below the second diagonal vanish, while all those in it are equal to $(n-1)!$, whence the value of the determinant is

$$(-1)^{\frac{n(n-1)}{2}} \{(n-1)!\}^n.$$

If m is less than $n - 1$ the determinant vanishes.

21. The determinant of order $r + 1$,

$$\begin{vmatrix} m_p, & m_{p+1}, & m_{p+2} & \dots & m_{p+r} \\ (m+1)_p, & (m+1)_{p+1}, & (m+1)_{p+2} & \dots & (m+1)_{p+r} \\ (m+2)_p, & (m+2)_{p+1}, & (m+2)_{p+2} & \dots & (m+2)_{p+r} \\ \dots & \dots & \dots & \dots & \dots \\ (m+r)_p, & (m+r)_{p+1}, & (m+r)_{p+2} & \dots & (m+r)_{p+r} \end{vmatrix},$$

where $$m_p = \frac{m(m-1)\dots(m-p+1)}{1 \cdot 2 \dots p},$$

though not orthosymmetrical, is of a similar nature; let us call it $V_{m,p}$.

Divide its first row by m, the second by $m+1$, ... its $(r+1)$th by $m+r$. Then multiply the first column by p, the second by $p+1$, ... the last by $p+r$. Then

$$V_{m,p} = \frac{m(m+1)\dots(m+r)}{p\,(p+1)\dots(p+r)} \times$$

$$\begin{vmatrix} (m-1)_{p-1}, & (m-1)_p & \dots (m-1)_{p+r-1} \\ m_{p-1}, & m_p & \dots m_{p+r-1} \\ \dots & \dots & \dots \\ (m+r-1)_{p-1}, & (m+r-1)_p & \dots (m+r-1)_{p+r-1} \end{vmatrix},$$

or, if we multiply numerator and denominator of the fraction by

$(r+1)!$,

$$V_{m,p} = \frac{(m+r)_{r+1}}{(p+r)_{r+1}} V_{m-1,\,p-1}.$$

Thus by giving to m and p different values we obtain the series of equations

$$V_{m-1,\,p-1} = \frac{(m+r-1)_{r+1}}{(p+r-1)_{r+1}} V_{m-2,\,p-2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$V_{m-p+1,\,1} = \frac{(m+r-p+1)_{r+1}}{(r+1)_{r+1}} V_{m-p,\,0}.$$

Now $V_{m-p,0}$ is the value of the last determinant in III. 5, when we write $m-p$ for m and 1 for d. Hence its value is unity, which gives, when we multiply the above equations together and cancel like factors,

$$V_{m,p} = \frac{(m+r)_{r+1}(m+r-1)_{r+1}\dots(m+r-p+1)_{r+1}}{(p+r)_{r+1}(p+r-1)_{r+1}\dots(r+1)_{r+1}}.$$

Another expression can be obtained for the determinant by dividing the first row by m_p, the second by $(m+1)_p$, ... the last by $(m+r)_p$. Then multiply the first column by p_0, the second by $(p+1)_1$, the last by $(p+r)_r$; the transformation gives

$$V_{m,p} = \frac{m_p(m+1)_p(m+2)_p\dots(m+r)_p}{p_p(p+1)_p(p+2)_p\dots(p+r)_p}.$$

A remarkable special case of the first form is when $p=1$, the value of the determinant being $(m+r)_{r+1}$, i.e. the last element in its leading diagonal.

22. If in the determinant of Art. 20

$$a_{k+1} = (c+k+m)_m = \frac{(c+k+m)(c+k+m-1)\dots(c+k+1)}{1.2\dots m},$$

then if $m = n-1$, $\Delta^{n-1}a_1 = 1$, and we have

$$\begin{vmatrix} (c+n-1)_{n-1}, & (c+n)_{n-1} & \dots (c+2n-2)_{n-1} \\ (c+n)_{n-1}, & (c+n+1)_{n-1} & \dots (c+2n-1)_{n-1} \\ \dots\dots & \dots\dots & \dots\dots \\ (c+2n-2)_{n-1}, & (c+2n-1)_{n-1} & \dots (c+3n-3)_{n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}}$$

23. Another class of determinants consists of those of the form

$$D = \begin{vmatrix} a_1, & a_2 & \dots & a_n \\ a_n, & a_1 & \dots & a_{n-1} \\ a_{n-1}, & a_n & \dots & a_{n-2} \\ \dots & \dots & \dots & \dots \\ a_2, & a_3 & \dots & a_1 \end{vmatrix};$$

where the element in the leading diagonal is always a_1, and the rest of the row is filled up with $a_2 \dots a_n$ in cyclical order.

The peculiar property of this determinant is that it divides by

$$a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1},$$

where ω is a root of the equation $x^n = 1$.

For if $A_1, A_2 \dots A_n$ are the complements of the elements of the first row of this determinant we have (IV. 11)

$$\left.\begin{aligned} a_1A_1 + a_2A_2 + \dots + a_nA_n &= D \\ a_1A_2 + a_2A_3 + \dots + a_nA_1 &= 0 \\ \dots\dots\dots\dots\dots\dots & \\ a_1A_n + a_2A_1 + \dots + a_nA_{n-1} &= 0 \end{aligned}\right\} \quad (1).$$

Now consider the product

$$(a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1})(A_1 + A_2\omega^{-1} + A_3\omega^{-2} + \dots + A_n\omega^{-n+1}).$$

The coefficient of ω^{k-1} is

$$A_1a_k + A_2a_{k+1} + \dots + A_na_{k-1}.$$

If k is equal to unity this is equal to D, by the first of equations (1), but if k is not unity it vanishes by one of the other equations. Thus D divides by

$$a_1 + a_2\omega + \dots + a_n\omega^{n-1}.$$

Hence

$$D = (a_1 + a_2 + \dots + a_n)\,\Pi\,(a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1}),$$

where ω is one of the roots of the equation $x^n - 1 = 0$, unity excepted.

24. Another elegant demonstration of the theorem of the preceding article is the following. If $\omega_1, \omega_2 \dots \omega_n$ are the n roots of unity, let

$$P = \begin{vmatrix} 1, & \omega_1, & \omega_1^2 & \dots & \omega_1^{n-1} \\ 1, & \omega_2, & \omega_2^2 & \dots & \omega_2^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ 1, & \omega_n, & \omega_n^2 & \dots & \omega_n^{n-1} \end{vmatrix}.$$

Then if we write

$$a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1} = \phi(\omega),$$

and remember that $\omega^n = 1$,

$$DP = \begin{vmatrix} \phi(\omega_1), & \phi(\omega_2) & \dots \\ \omega_1\phi(\omega_1), & \omega_2\phi(\omega_2) & \dots \\ \omega_1^2\phi(\omega_1), & \omega_2^2\phi(\omega_2) & \dots \\ \dots & \dots & \dots \\ \omega_1^{n-1}\phi(\omega_1), & \omega_2^{n-1}\phi(\omega_2) & \dots \end{vmatrix}$$

$$= P\phi(\omega_1)\,\phi(\omega_2) \dots \phi(\omega_n),$$

whence $$D = \phi(\omega_1)\,\phi(\omega_2) \dots \phi(\omega_n).$$

25. Mr Glaisher has shewn that a determinant, such as that in Art. 23, of order $2n$, can be expressed as a similar determinant of order n. Namely

$$\begin{vmatrix} a_1, & a_2 & \dots & a_{2n} \\ a_{2n}, & a_1 & \dots & a_{2n-1} \\ a_{2n-1}, & a_{2n} & \dots & a_{2n-2} \\ \dots & \dots & \dots & \dots \\ a_2, & a_3 & \dots & a_1 \end{vmatrix} = \begin{vmatrix} A_1, & A_2 & \dots & A_n \\ A_n, & A_1 & \dots & A_{n-1} \\ \dots & \dots & \dots & \dots \\ A_2, & A_3 & \dots & A_1 \end{vmatrix},$$

where

$$A_1 = a_1a_1 - a_{2n}a_2 + a_{2n-1}a_3 \dots - a_2a_{2n}$$
$$A_2 = a_3a_1 - a_2a_2 + a_1a_3 \dots - a_4a_{2n}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$A_r = a_{2r-1}a_1 - a_{2r-2}a_2 + a_{2r-3}a_3 \dots - a_{2r}a_{2n}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

For the first determinant

$$= \Pi (a_1 + a_2\omega + a_3\omega^2 + \dots + a_{2n}\omega^{2n-1}),$$

ω being a $2n$th root of unity; and since for every root ω there is a root $-\omega$, this

$$= \Pi (A_1 + A_2\omega^2 + A_3\omega^4 + \dots + A_n\omega^{2n-2}),$$

which product is equal to the second determinant. For the $2n$th roots of unity being denoted by $\pm 1, \pm \omega_1, \pm \omega_2 \dots \pm \omega_{n-1}$, the nth roots of unity are $1, \omega_1^2, \omega_2^2 \dots \omega^2_{n-1}$.

For example, if $n = 2$,

$$\begin{vmatrix} a, & b, & c, & d \\ d, & a, & b, & c \\ c, & d, & a, & b \\ b, & c, & d, & a \end{vmatrix} = \begin{vmatrix} A, & B \\ B, & A \end{vmatrix},$$

where
$$A = a^2 + c^2 - 2bd,$$
$$B = -b^2 - d^2 + 2ac,$$

and the value of the determinant is

$$a^4 - b^4 + c^4 - d^4 - 2a^2c^2 + 2b^2d^2 - 4a^2bd + 4b^2ac - 4c^2bd + 4d^2ac.$$

26. If in the determinant of Art. 23 we suppose

$$a_r = \frac{x^{r-1}}{(r-1)!} + \frac{x^{n+r-1}}{(n+r-1)!} + \frac{x^{2n+r-1}}{(2n+r-1)!} + \dots$$

$$D = \epsilon^x \Pi (a_1 + a_2\omega + a_3\omega^2 + \dots + a_n\omega^{n-1})$$
$$= \epsilon^x \Pi\, \epsilon^{\omega x}$$
$$= \epsilon^{x(1+\omega_1+\omega_2+\dots+\omega_{n-1})}$$
$$= 1.$$

27. Determinants whose elements are binomial coefficients have been discussed with great minuteness by v. Zeipel, who has given an immense number of theorems relating to this class of determinants. One or two of these we shall now consider.

The value of the determinant

$$\begin{vmatrix} m_k, & n, & pm_1, & qm_2 & \dots & tm_{k-1} \\ (m+1)_k, & n+1, & (p+1)(m+1)_1, & (q+1)(m+1)_2 & \dots & (t+1)(m+1)_{k-1} \\ (m+2)_k, & n+2, & (p+2)(m+2)_1, & (q+2)(m+2)_2 & \dots & (t+2)(m+2)_{k-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (m+k)_k, & n+k, & (p+k)(m+k)_1, & (q+k)(m+k)_2 & \dots & (t+k)(m+k)_{k-1} \end{vmatrix}$$

is $$(m-n)(m-p-1)(m-q-2)\dots(m-t-k+1).$$

We must first shew that the determinant vanishes when m is equal to any one of the quantities

$$n, \ p+1, \ q+2 \dots t+k-1.$$

First let $m=n$, then the determinant is

$$\begin{vmatrix} m_k, & m, & pm_1, & qm_2 & \dots \\ (m+1)_k, & m+1, & (p+1)(m+1)_1, & (q+1)(m+1)_2 & \dots \\ \dots & \dots & \dots & \dots & \dots \\ (m+k)_k, & m+k, & (p+k)(m+k)_1, & (q+k)(m+k)_2 & \dots \end{vmatrix}.$$

If we subtract the second column, multiplied by p, from the third we see that the determinant is independent of p. Do this, and divide the first row by m, the second by $m+1$, the third by $m+2\dots$, then multiply the first column by k, the fourth by 2, the fifth by 3 ..., then the determinant reduces to the product of

$$\frac{m(m+1)(m+2)\dots(m+k)}{1.2\dots k}$$

and the determinant

$$\begin{vmatrix} (m-1)_{k-1}, & 1, & 0, & q(m-1)_1, & r(m-1)_2 & \dots \\ m_{k-1}, & 1, & 1, & (q+1)m_1, & (r+1)m_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (m+k-1)_{k-1}, & 1, & k, & (q+k)(m+k-1)_1, & (r+k)(m+k-1)_2 & \dots \end{vmatrix}$$

Multiply the second column by $q(m-1)_1$, the third by

$$q(m-1)_0 + 1 \, . \, m_1,$$

and subtract their sum from the fourth column, and we get the new determinant

$$2\begin{vmatrix} (m-1)_{k-1}, & 1, & 0, & 0, & r(m-1)_2 & \dots \\ m_{k-1}, & 1, & 1, & 0, & (r+1)m_2 & \dots \\ (m+1)_{k-1}, & 1, & 2, & 1, & (r+2)(m+1)_2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ (m+k-1)_{k-1}, & 1, & k_1, & k_2, & (r+k)(m+k-1)_2 & \dots \end{vmatrix}.$$

In this determinant multiply the second column by $r(m-1)_2$, the third by $r(m-1)_1 + 1 \,.\, m_2$, the fourth by $r(m-1)_0 + 2 \,.\, m_1$, and subtract the sum of their elements so multiplied from the elements of the fifth column, and proceed in a similar way with the altered determinant. Finally we reduce the determinant to the product of a finite number of factors and

$$\begin{vmatrix} (m-1)_{k-1}, & 1, & 0, & 0 & \dots & 0, & 0 \\ m_{k-1}, & 1, & 1, & 0 & \dots & 0, & 0 \\ (m+1)_{k-1}, & 1, & 2, & 1 & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (m+k-1)_{k-1}, & 1, & k_1, & k_2 & \dots & k_{k-2}, & k_{k-1} \end{vmatrix}.$$

In this determinant multiply the second column by $(m-1)_{k-1}$, the third by $(m-1)_{k-2}$, the fourth by $(m-1)_{k-3}$, &c., and subtract their sum from the elements of the first column; then each element of the first column, and consequently the determinant vanishes. Hence our determinant divides by $m-n$. Similarly we can shew that it divides by each of the other factors, hence it is equal to

$$C(m-n)(m-p-1)(m-q-2)\dots(m-t-k+1),$$

where C is independent of $n, p, q \dots t$, because the determinant is linear in each of these quantities.

To find the value of C put

$$n = p = q = \dots = t = 0;$$

then we get

$$m_k \begin{vmatrix} 1, & (m+1)_1, & (m+1)_2 & \dots \\ 2, & 2(m+2)_1, & 2(m+2)_2 & \dots \\ 3, & 3(m+3)_1, & 3(m+3)_2 & \dots \\ \dots & \dots & \dots & \dots \\ k, & k(m+k)_1, & k(m+k)_2 & \dots \end{vmatrix}$$

$$= Cm(m-1)\dots(m-k+1).$$

But the determinant last written is equal to $k!$, as we see by putting $d=1$ in the last determinant of III. 5. Hence

$$C=1;$$

thus the theorem is proved.

28. The determinant

$$\begin{vmatrix} m_k, & n, & pm_1 & \dots\ qm_{k-1},\ sm_k,\ \dots & um_{r-1} \\ (m+1)_k, & n+1, & (p+1)(m+1)_1\dots & \dots\dots\dots & (u+1)(m+1)_{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ (m+k)_k, & n+k, & (p+k)(m+k)_1\dots & \dots\dots\dots & (u+k)(m+k)_{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ (m+r)_k, & n+r, & (p+r)(m+r)_1\dots & \dots\dots\dots & (u+r)(m+r)_{r-1} \end{vmatrix}$$

is equal to the product of

$$(k+1)(k+2)\dots r$$

and

$$\begin{vmatrix} m_k, & n, & pm_1, & \dots & qm_{k-1} \\ (m+1)_k, & n+1, & (p+1)(m+1)_1 & \dots & (q+1)(m+1)_{k-1} \\ \dots & \dots & \dots & \dots & \dots \\ (m+k)_k, & n+k, & (p+k)(m+k)_1 & \dots & (q+k)(m+k)_{k-1} \end{vmatrix} \quad (1).$$

That is to say, it is independent of the $r-k$ quantities $s, \dots u$.

To prove this, apply to the determinant the operations of III. 5 (iv). Then in place of any element P in the jth row we must write

$$\Delta^{j-1}P.$$

Then in the first column every element after the $(k+1)$th vanishes, in each of the others every element below the leading diagonal vanishes, while the element of the ith column which is in the leading diagonal is $(i-1)$.

Hence if we expand the determinant by Laplace's theorem, according to minors of the first k columns, it reduces to

$$(k+1)(k+2)\dots r \begin{vmatrix} m_k, & n, & pm_1 & \dots\ qm_{k-1} \\ m_{k-1}, & 1, & [pm_0+(m+1)_0] & \dots \\ m_{k-2}, & 0, & 2(m+1)_0 & \dots \\ \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots \end{vmatrix},$$

which proves the theorem. For the last determinant is the result of operating, as in III. 5 (iv), on the determinant (1). The determinant (1) is known by Art. 27, and hence we know the value of the new determinant.

29. Next let us consider

$$\begin{vmatrix} m_k, & nm_d, & pm_{d+1} & \dots & um_{d+r-1} \\ (m+1)_k, & (n+1)(m+1)_d, & (p+1)(m+1)_{d+1} \dots & & \dots \\ \dots & \dots & \dots & & \dots \\ (m+r)_k, & (n+r)(m+r)_d, & (p+r)(m+r)_{d+1} \dots & & \dots \end{vmatrix};$$

where k has any value from d to $d+r-1$ inclusive.

Divide the rows by

$$m_d, \ (m+1)_d \dots (m+r)_d$$

respectively, and multiply the columns by

$$k_{k-d}, \ 1, \ (d+1)_1, \ (d+2)_2 \dots$$

Then our determinant is equal to

$$\frac{m_d (m+1)_d (m+2)_d \dots (m+r)_d}{k_{k-d}(d+1)_1 (d+2)_2 \dots (d+r-1)_{r-1}} \dots\dots\dots\dots (1)$$

multiplied by the determinant

$$\begin{vmatrix} (m-d)_{k-d}, & n, & p(m-d)_1 & \dots \\ (m-d+1)_{k-d}, & n+1, & (p+1)(m-d+1)_1 & \dots \\ \dots & \dots & \dots & \dots \\ (m-d+r)_{k-d}, & n+r, & (p+r)(m-d+r)_1 & \dots \end{vmatrix},$$

which by the preceding articles is equal to

$$(k-d+1)(k-d+2)\dots r(m-d-n)(m-d-p-1)$$
$$(m-d-q-2)\dots \qquad (2),$$

being independent of the last $(r-k+d)$ of the quantities n, $p \dots u$.

The determinant we started with is equal to the product of (1) and (2).

30. In the determinant of the last article let

$$n = p = \ldots = u = \tfrac{1}{2}, \quad k = d = 1\,;$$

then, if we multiply both sides by 2^r, we obtain

$$\begin{vmatrix} m_1, & m_1, & m_2 & \ldots & m_r \\ (m+1)_1, & 3\,(m+1)_1, & 3\,(m+1)_2 & \ldots & 3\,(m+1)_r \\ (m+2)_1, & 5\,(m+2)_1, & 5\,(m+2)_2 & \ldots & 5\,(m+2)_r \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ (m+r)_1, & (2r+1)\,(m+r)_1, & (2r+1)\,(m+r)_2 & \ldots & (2r+1)\,(m+r)_r \end{vmatrix}$$

$$= 2^r m\,(m+1)\ldots(m+r).$$

Divide both sides by $m\,(m+1)\ldots(m+r)$, and then multiply both sides by $r!$, thus

$$\begin{vmatrix} 1, & 1, & (m-1)_1, & \ldots & (m-1)_{r-1} \\ 1, & 3, & 3m_1, & \ldots & 3\,(m)_{r-1} \\ 1, & 5, & 5\,(m+1)_1 & \ldots & 5\,(m+1)_{r-1} \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 2\,.\,4\,.\,6\ldots 2r.$$

Hence, changing $m-1$ into m, if we write

$$u_r = \begin{vmatrix} 1, & 1, & m_1, & m_2 & \ldots & m_r \\ \frac{1}{3}, & 1, & (m+1)_1, & (m+1)_2 & \ldots & (m+1)_r \\ \frac{1}{5}, & 1, & (m+2)_1, & (m+2)_2 & \ldots & (m+2)_r \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix},$$

we have by Wallis' theorem

$$\text{Lim.}\ (2r+1)\,u^2_{r-1} = \frac{\pi}{2},$$

when r, and therefore the order of the determinant, is infinite.

CHAPTER IX.

ON CUBIC DETERMINANTS AND DETERMINANTS WITH MULTIPLE SUFFIXES.

1. JUST as when n^2 elements are given we can arrange them in the form of a square, so when n^3 elements are given we can arrange them in the form of a cube. Then we can indicate the position of the elements by means of three suffixes. The elements will lie in three sets of parallel planes; supposing the cube containing the elements to stand on a table with one face towards us, we may for convenience call those planes parallel to the face on which the cube rests strata, those parallel to the face in front of us planes, and the perpendicular planes sections.

2. An element of such an array will be denoted by a_{ijk}, where the suffixes mean that it stands in the ith stratum, jth plane, and kth section.

The set of elements in the leading diagonal will be

$$a_{111} a_{222} \dots a_{nnn}.$$

From this we can form a function analogous to a determinant, and hence called a cubic determinant, by the following process.

From the leading term $a_{111} a_{222} \dots a_{nnn}$ we form $n!$ terms by writing for the series of third suffixes all possible permutations of $1, 2 \dots n$, giving to each of these terms a sign corresponding to the class of the permutation. Then from each of the terms so obtained we derive $n!$ new terms by writing for the series of second suffixes all possible permutations of $1, 2 \dots n$, giving to each new term, relatively to the term from which it is derived,

the sign corresponding to the class of the permutation. The sum of all these $\{n!\}^2$ terms is called a cubic determinant, and is denoted by

$$\Sigma \pm a_{111} a_{222} \dots a_{nnn},$$

or by

$$|a_{ijk}| \quad (i, j, k = 1, 2, \dots n).$$

3. Just as an ordinary determinant can be represented as the product of n alternate numbers, so a cubic determinant can be represented as the product of n factors lineo-linear in two sets of alternate units.

If $e_1, e_2 \dots e_n$; $\epsilon_1, \epsilon_2 \dots \epsilon_n$ are two independent sets of alternate units, then the determinant of Art. 2 is equal to the product

$$\begin{aligned}\Pi \{&a_{i11}\epsilon_1 e_1 + a_{i12}\epsilon_1 e_2 + \dots + a_{i1n}\epsilon_1 e_n \\ &+ a_{i21}\epsilon_2 e_1 + a_{i22}\epsilon_2 e_2 + \dots + a_{i2n}\epsilon_2 e_n \\ &+ \dots\dots\dots\dots\dots\dots \\ &+ a_{in1}\epsilon_n e_1 + a_{in2}\epsilon_n e_2 + \dots + a_{inn}\epsilon_n e_n\}\end{aligned} \quad (i = 1, 2 \dots n).$$

For if we consider any term of the product, it will vanish if it contains two ϵ's or two e's with the same suffix, i.e. if two a's with like second or third suffix occur in the term, which ensures that all terms which do not belong to the determinant vanish. Thus every term which does not vanish contains some permutation of the units $\epsilon_1, \epsilon_2 \dots \epsilon_n$ and $e_1, e_2 \dots e_n$ as a factor, and if the units be brought to this order the sign of the term will be $(-1)^{\mu+\nu}$; where μ is the number of inversions in the ϵ's, i.e. in the second suffixes of the term, and ν the like number for the e's or third suffixes. That is to say each term of the product is a term of the determinant with its proper sign. Thus the determinant is correctly represented by the product.

Just as an ordinary determinant is the product of linear functions of the elements of a row, a cubic determinant is the product of linear functions of the elements of a stratum.

By means of this representation we can deduce the properties of cubic determinants.

4. The sign of the determinant is changed if we interchange two planes or sections.

For interchanging two planes is the same thing as inter-

changing two ϵ's, and interchanging two sections the same as interchanging two e's. Either of these changes alters the sign of every term, and therefore of the whole determinant.

5. Interchanging two strata does not alter the sign of the determinant.

For we can represent the determinant by either of the two products

$$\left.\begin{array}{l}\Pi\,(b_{i1}e_1 + b_{i2}e_2 + \ldots + b_{in}e_n)\\ \Pi\,(c_{i1}\epsilon_1 + c_{i2}\epsilon_2 + \ldots + c_{in}\epsilon_n)\end{array}\right\}\ (i = 1,\ 2 \ldots n)$$

where
$$b_{ik} = a_{i1k}\epsilon_1 + a_{i2k}\epsilon_2 + \ldots + a_{ink}\epsilon_n$$
$$c_{ik} = a_{ik1}e_1 + a_{ik2}e_2 + \ldots + a_{ikn}e_n.$$

From the first form we see that the determinant, on interchanging two strata, suffers a change of sign as being the product of alternate numbers belonging to the system e; from the second we see that it also suffers independently a change of sign as being the product of alternate numbers belonging to the system ϵ. Thus on interchanging two strata the determinant undergoes two changes of sign, and hence remains unaltered.

6. A cubic determinant of order n is the sum of $n!$ ordinary determinants, each of order n.

For as in Art. 5

$$A = \Pi\,(c_{i1}\epsilon_1 + c_{i2}\epsilon_2 + \ldots + c_{in}\epsilon_n)$$

where c_{ik} has the same meaning as in Art. 5. Hence, by I. 19,

$$A = |c_{ik}|.$$

Thus the cubic determinant is equal to an ordinary determinant of the same order, whose elements are alternate numbers. To split up this determinant into others with simple elements we must take a partial column from each column of the determinant, but if we take a partial column in the pth place from one column we cannot take a partial column in the pth place from any other column, for then e_p would occur twice, and the corresponding determinant must vanish. Hence each selection of partial columns must be a permutation of 1, 2 … n, there are $n!$ such selections, and as many determinants with simple elements.

Thus $$A = \Sigma\, |a_{ij(k)}|,$$
where the determinant on the right is an ordinary determinant; k is put in brackets to remind us that though it varies from one column to another, in the same determinant it remains fixed. This theorem is also an obvious consequence of Art. 2.

7. If in the preceding article we suppose all the first suffixes to be the same, all the determinants on the right would become alike, only their columns being permuted, and each determinant would have the sign corresponding to that permutation, hence suppressing the first suffixes altogether, the cubic determinant is now equal to
$$(n\,!)\,|a_{jk}| \qquad (j,\ k = 1,\ 2 \ldots n).$$

This then is the value of the cubic determinant whose strata consist of the determinant
$$\begin{vmatrix} a_{11} \ldots a_{1n} \\ \cdots\cdots\cdots \\ a_{n1} \ldots a_{nn} \end{vmatrix}$$
repeated n times.

8. The product of two ordinary determinants, each of order n, may be expressed as a cubic determinant of order n.

Let $$A = |a_{ik}| = A_1 A_2 \ldots A_n,$$
$$B = |b_{ik}| = B_1 B_2 \ldots B_n,$$
where $$A_i = a_{i1}\epsilon_1 + a_{i2}\epsilon_2 + \ldots + a_{in}\epsilon_n,$$
$$B_i = b_{i1}e_1 + b_{i2}e_2 + \ldots + b_{in}e_n,$$
the systems of units e and ϵ being independent.

Then $$\begin{aligned} AB &= \Pi A_i B_i \\ &= \Pi\,(a_{i1}b_{i1}\epsilon_1 e_1 + a_{i1}b_{i2}\epsilon_1 e_2 + \ldots + a_{i1}b_{in}\epsilon_1 e_n \\ &\quad + a_{i2}b_{i1}\epsilon_2 e_1 + a_{i2}b_{i2}\epsilon_2 e_2 + \ldots + a_{i2}b_{in}\epsilon_2 e_n \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + a_{in}b_{i1}\epsilon_n e_1 + a_{in}b_{i2}\epsilon_n e_2 + \ldots + a_{in}b_{in}\epsilon_n e_n). \end{aligned}$$

Now if $$c_{ijk} = a_{ij}b_{ik},$$
the product on the right is the cubic determinant of the elements c_{ijk}. Thus the theorem is proved.

By multiplying A_i and B_i together we avoided any inversion of the A's and B's among themselves. If we allow for the consequent changes of sign we can have as many such inversions as we please, and so vary the form of the cubic determinant which represents the product.

9. The product of a cubic determinant A, whose elements are a_{ijk}, and of an ordinary determinant B, whose elements are b_{ik}, is expressible as a cubic determinant C, whose elements are c_{ijk}, where

$$c_{ijk} = b_{j1} a_{ik1} + b_{j2} a_{ik2} + \dots + b_{jn} a_{ikn}.$$

To prove this, we observe that

$$\begin{aligned} C = \Pi\, (&c_{i11} \epsilon_1 e_1 + c_{i12} \epsilon_1 e_2 + \dots + c_{i1n} \epsilon_1 e_n \\ &+ c_{i21} \epsilon_2 e_1 + c_{i22} \epsilon_2 e_2 + \dots + c_{i2n} \epsilon_2 e_n \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &+ c_{in1} \epsilon_n e_1 + c_{in2} \epsilon_n e_2 + \dots + c_{inn} \epsilon_n e_n) \\ = \Pi\, (&a_{i11} B_1 e_1 + a_{i12} B_1 e_2 + \dots + a_{i1n} B_1 e_n \\ &+ a_{i21} B_2 e_1 + a_{i22} B_2 e_2 + \dots \\ &+ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots); \end{aligned}$$

where $\quad B_j = b_{1j} \epsilon_1 + b_{2j} \epsilon_2 + \dots + b_{nj} \epsilon_n.$

Now since the alternate numbers B_j follow the same laws as units, this last product is a representation of the cubic determinant A by means of the units e and B. Thus

$$\begin{aligned} C &= A \,.\, e_1 \dots e_n \,.\, B_1 \dots B_n \\ &= AB. \end{aligned}$$

10. It is now an obvious step to construct functions analogous to determinants by means of letters with more than three suffixes, though when we take elements with more than three suffixes we cease to be able to picture to ourselves their arrangement topographically as we can in the case of elements with one, two or three suffixes. We can, however, conceive a set of elements with p suffixes such as

$$a_{ijk\dots l},$$

n^p in number, to be arranged in p sets of rectangular planes in a space of p dimensions, and forming a rectangular parallelo-

scheme of p dimensions. (Cf. Schläfli, *Quarterly Jour.* II. p. 278.) The elements which have all suffixes the same, except i, lie in the same line, those which have all suffixes the same, with the exception of i and j, lie in the same plane, ... those which have only l in common lie in a rectangular paralleloscheme of $p-1$ dimensions.

The product of the elements

$$a_{11\dots 1}a_{22\dots 2}\dots a_{nn\dots n}$$

is called the leading term of the determinant of the pth class, which is formed by keeping the first suffixes unaltered, and writing for each set of the other suffixes all possible permutations of $1, 2 \dots n$. To each term so obtained we give the sign corresponding to the sum of the number of inversions in the $p-1$ sets of variable suffixes.

The whole number of terms is $\{n!\}^{p-1}$.

11. The determinant of the pth class can be represented as a product of linear factors of the elements which lie in the same paralleloscheme of $p-1$ dimensions.

If
$$\begin{array}{c} e_1,\ e_2 \dots e_n \\ \epsilon_1,\ \epsilon_2 \dots \epsilon_n \\ \dots\dots\dots\dots \\ \eta_1,\ \eta_2 \dots \eta_n \end{array}$$
be $p-1$ sets of alternate units; it is plain from reasoning similar to that in Art. 3, that the function

$$A = \Pi\Sigma a_{ijk\dots l}e_j\epsilon_k \dots \eta_l$$

(where the sum is formed by giving to each of the suffixes $j, k \dots l$ all values from 1 to n, and then forming the product of such sums for the values $1, 2 \dots n$ of i) is a determinant of the pth class and nth order, such as we have defined in Art. 10.

12. This definition is strictly analogous to those for determinants of the second and third class. A determinant of the second class is the product of linear functions of the elements of a row, one of the third class the product of n factors linear in the elements of a stratum. Here the determinant of the pth class is the product of n factors linear in the elements of a paralleloscheme of $p-1$ dimensions.

13. It is clear that by the interchange of any two suffixes, except the first, the determinant changes sign. Also since the factors of the determinant can be written as linear expressions of each of the $p-1$ sets of alternate units, it follows that by the interchange of two first suffixes the determinant undergoes $p-1$ independent changes of sign. Thus the determinant remains unaltered or changes sign according as its class is odd or even.

14. We have kept the first suffixes in their natural order. It is however indifferent which set of suffixes is retained fixed. If the class of the determinant is odd, it is perhaps more symmetrical to keep the middle suffix unaltered; the determinant is however not the same as before.

15. The product of a cubic determinant A, whose elements are a_{ijk}, and of an ordinary determinant B, whose elements are b_{ik}, can be represented as a determinant of the fourth class C, whose elements c_{ijkl} are given by

$$c_{ijkl} = a_{ijk} b_{il}.$$

For
$$A = \Pi\,(a_{i11}\epsilon_1 e_1 + a_{i12}\epsilon_1 e_2 + \ldots + a_{i1n}\epsilon_1 e_n + a_{i21}\epsilon_2 e_1 + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots),$$
$$B = \Pi\,(b_{i1}\eta_1 + b_{i2}\eta_2 + \ldots + b_{in}\eta_n).$$

Thus clearly

$$AB = \Pi\,(\Sigma c_{ijkl}\epsilon_j e_k \eta_l) \qquad \begin{array}{l}(\text{In } \Sigma\ j,\ k,\ l = 1,\ 2 \ldots n)\\ (\text{In } \Pi\ i = 1,\ 2 \ldots n)\end{array}$$

which proves the theorem.

16. The product of two cubic determinants A and B, whose elements are a_{ijk} and b_{ijk}, both of order n, can be represented either as a determinant of the fifth class, whose elements are

$$c_{ipqrs} = a_{ipq} b_{irs},$$

or as a determinant of the fourth class, whose elements are given by

$$c_{ijkl} = \Sigma a_{pij} b_{pkl} \qquad (p = 1,\ 2 \ldots n);$$

the order of both determinants being n.

The first part of the theorem is proved as follows:

$$A = \Pi \Sigma a_{ipq}\epsilon_p e_q.$$

$$(\text{In } \Sigma\ p,\ q = 1,\ 2 \ldots n;\ \text{ in } \Pi\ i = 1,\ 2 \ldots n.)$$

$$B = \Pi\Sigma b_{irs} j_r k_s.$$

(In Σ $r, s = 1, 2 \dots n$; in Π $i = 1, 2 \dots n$.)

Thus
$$AB = \Pi\Sigma a_{ipq} b_{irs} \epsilon_p e_q j_r k_s$$
$$= \Pi\Sigma c_{ipqrs} \epsilon_p e_q j_r k_s.$$

(In Σ $p, q, r, s = 1, 2 \dots n$; in Π $i = 1, 2 \dots n$.)

Which by definition proves the theorem.

For the second part of the theorem we have
$$C = \Pi\Sigma c_{ijkl} e_j \epsilon_k \eta_l.$$

Now the sum under the product sign
$$= \Sigma e_j \{a_{ij1} B_1 + a_{ij2} B_2 + \dots + a_{ijn} B_n\} \qquad (j = 1, 2 \dots n),$$
where
$$B_p = b_{p11} \epsilon_1 \eta_1 + b_{p12} \epsilon_1 \eta_2 + \dots + b_{p1n} \epsilon_1 \eta_n$$
$$+ b_{p21} \epsilon_2 \eta_1 + b_{p22} \epsilon_2 \eta_2 + \dots + b_{p2n} \epsilon_2 \eta_n$$
$$+ \dots$$
and if we write
$$A_{iq} = a_{i1q} e_1 + a_{i2q} e_2 + \dots + a_{inq} e_n$$
the sum becomes
$$B_1 A_{i1} + B_2 A_{i2} + \dots + B_n A_{in}.$$

The product of this has to be taken for all values of i. It must always be taken so that in each term we have the product $B_1 B_2 \dots B_n$; for if two B's are repeated the term vanishes. The value of this product is B.

The remaining factors in the term are
$$A_{1p} A_{2q} \dots A_{nr},$$
where $p, q \dots r$ is a permutation of $1, 2 \dots n$. This is an ordinary determinant of class 2. Comparing this with Art. 6, we see that it is a term in the expansion of the cubic determinant A as a sum of determinants of class 2. All these terms occur in our product. Thus
$$C = A \,.\, B.$$

17. The following theorem regarding the product of two determinants of any class can be proved by the preceding methods.

The product of two determinants of classes p and q, whose elements are $a_{ij\ldots l}$ and $b_{ij\ldots k}$ respectively, can be represented either as a determinant of class $p+q-1$, whose elements are

$$c_{ij\ldots luv\ldots s} = a_{ij\ldots l} b_{iu\ldots s},$$

or as a determinant of class $p+q-2$, whose elements are

$$c_{j\ldots luv\ldots s} = \Sigma a_{ij\ldots l}\, b_{iu\ldots s} \qquad (i = 1, 2 \ldots n),$$

all the determinants being of order n.

18. It is not difficult to see how the theorems with regard to determinants of the second class (i.e. ordinary determinants) can be extended to determinants of any other class. It is probable that determinants of higher class possess many properties peculiar to themselves, though as yet not many of these have been investigated. The complement of any element of a determinant is a determinant of the same class and next lower order. The extension of Laplace's theorem would shew how a determinant of class p and order n could be expanded in a series of products of pairs of determinants of class p and orders m and $n-m$.

19. There is no difficulty in writing down the expansions of determinants of any required class or order. The number of terms however increases very rapidly.

The following are the expansions of determinants of the second order, and classes 3 and 4 respectively:

$$\Sigma \pm (111)(222) = (111)(222) - (121)(212) + (122)(211) - (112)(221)$$

$$\begin{aligned} \Sigma \pm (1111)(2222) = {} & (1111)(2222) - (1112)(2221) + (1212)(2121) \\ & - (1211)(2122) + (1122)(2211) - (1121)(2212) \\ & + (1221)(2112) - (1222)(2111), \end{aligned}$$

while for the determinant of class 3 and order 3,

$$\begin{aligned} \Sigma \pm (111)(222)(333) = {} & (111)(222)(333) - (121)(212)(333) \\ & - (111)(232)(323) + (131)(212)(323) \\ & + (121)(232)(313) - (131)(222)(313) \\ & - (112)(221)(333) + (122)(211)(333) \\ & + (112)(231)(323) - (132)(211)(323) \\ & - (122)(231)(313) + (132)(221)(313) \\ & - (111)(223)(332) + (121)(213)(332) \\ & + (111)(233)(322) - (131)(213)(322) \\ & - (121)(233)(312) + (131)(223)(312) \\ & + (113)(221)(332) - (123)(211)(332) \end{aligned}$$

$$
\begin{aligned}
&-(113)(231)(322)+(133)(211)(322)\\
&+(123)(231)(312)-(133)(221)(312)\\
&+(112)(223)(331)-(122)(213)(331)\\
&-(112)(233)(321)+(132)(213)(321)\\
&+(122)(233)(311)-(132)(223)(311)\\
&-(113)(222)(331)+(123)(212)(331)\\
&+(113)(232)(321)-(133)(212)(321)\\
&-(123)(232)(311)+(133)(222)(311).
\end{aligned}
$$

20. We shall conclude this chapter with the following general theorems.

A determinant of any class, all of whose elements are equal to a, except those in the leading diagonal which are equal to x, is equal to
$$\{x+(n-1)\,a\}\,(x-a)^{n-1},$$
n being the order of the determinant.

We shall prove this for a cubic determinant, but the method is perfectly general.

$$
\begin{aligned}
D = \Pi\,(&ae_1\epsilon_1 + ae_1\epsilon_2 + \dots\\
&+ ae_2\epsilon_1 + ae_2\epsilon_2 + \dots\\
&+ \dots \quad + xe_i\epsilon_i + \dots)\\
= \Pi\,&\{aEE' + (x-a)\,e_i\epsilon_i\},
\end{aligned}
$$

where $\quad E = e_1 + e_2 + \dots + e_n, \quad E' = \epsilon_1 + \epsilon_2 + \dots + \epsilon_n.$

Hence, since E and E' are alternate numbers, any term in which they occur more than once vanishes.

Hence $\quad D = (x-a)^n + a\,(x-a)^{n-1}\,\Sigma\,\{EE'\Pi e_k\epsilon_k\}$
$$(k = 1,\ 2 \dots i-1,\ i+1 \dots n);$$

$$
\begin{aligned}
\therefore\ D &= (x-a)^n + na\,(x-a)^{n-1}\\
&= \{x+(n-1)\,a\}\,(x-a)^{n-1};
\end{aligned}
$$

for
$$
\begin{aligned}
Ee_1 \dots e_{i-1}e_{i+1} \dots e_n &= e_ie_1 \dots e_{i-1}e_{i+1} \dots e_n\\
&= (-1)^{i-1}e_1e_2 \dots e_n;
\end{aligned}
$$

and so $\quad E'\epsilon_1 \dots \epsilon_{i-1}\epsilon_{i+1} \dots \epsilon_n = (-1)^{i-1}\epsilon_1\epsilon_2 \dots \epsilon_n.$

The last theorem of IV. 25 can also be extended to determinants of higher class. For a cubic determinant we may state it as follows: If all the elements in the ith stratum are equal to a_i, with the exception of that which lies in the leading diagonal, whose value is x_i, then the value of the determinant is
$$f + \Sigma a_r f'(x_r)$$
with the notation given in IV. 25.

CHAPTER X.

DETERMINANTS OF INFINITE ORDER.

1. If in the symbol a_{ik} we suppose the suffixes i, k to assume independently all positive and negative integral values, including zero, we obtain a doubly infinite system of elements, which may be arranged according to the scheme

$$\begin{array}{ccccccc}
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & a_{-2,-2} & a_{-2,-1} & a_{-2,0} & a_{-2,1} & a_{-2,2} & \cdots \\
\cdots & a_{-1,-2} & a_{-1,-1} & a_{-1,0} & a_{-1,1} & a_{-1,2} & \cdots \\
\cdots & a_{0,-2} & a_{0,-1} & a_{0,0} & a_{0,1} & a_{0,2} & \cdots \\
\cdots & a_{1,-2} & a_{1,-1} & a_{1,0} & a_{1,1} & a_{1,2} & \cdots \\
\cdots & a_{2,-2} & a_{2,-1} & a_{2,0} & a_{2,1} & a_{2,2} & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots
\end{array}$$

Thus if we take a pair of rectangular axes, and measure ordinates positively from left to right, and abscissæ positively from above downwards, the position of a_{ik} will be at the point (i, k).

2. Let p, q be positive integers, and suppose that

$$p + q = m;$$

then if i and k each range from $-p$ to q the determinant $|a_{ik}|$ is of the mth order and its elements are in the same relative positions as in the infinite array above indicated.

Let m increase indefinitely in such a way that p, q both become indefinitely large: then the ultimate behaviour of $|a_{ik}|$ is analogous to that of an infinite series. It may become infinite; it may be indeterminate; it may converge. The case with which we shall deal almost exclusively is that in which the determinant converges to a definite limit, A, which is independent of the way in which p, q become infinite. We may then, for simplicity,

suppose that i, k each range from $-n$ to $+n$, where n is an integer which ultimately increases without limit. Thus A is the limit of the sequence

$$A_0,\ A_1,\ A_2, \dots A_n, \dots$$

where $A_0 = a_{0,0}$ and

$$A_n = |a_{ik}| \qquad (i, k = -n \dots +n).$$

The element $a_{0,0}$ may be called the central element, and

$$\dots a_{-1,-1},\ a_{0,0},\ a_{1,1} \dots$$

the diagonal elements of the infinite determinant A.

Any diagonal element may be taken as the central element; because if the notation is changed by writing

$$a_{i+\lambda, k+\lambda} = b_{i,k}$$

where λ is any fixed positive or negative integer, the sequence which in the new notation is B_0, B_1, B_2, etc. is simply one of the sequences which may be chosen to specify A.

3. If, in A, any two rows or columns at a finite distance from the centre are interchanged, the value of the new determinant is $-A$. For if we take n so large that A_n includes both the lines which are interchanged, the sequence

$$A_n,\ A_{n+1},\ A_{n+2}, \dots$$

becomes

$$-A_n,\ -A_{n+1},\ -A_{n+2}, \dots$$

the limit of which is $-A$.

In the same manner it can be proved that if two rows or columns are identical or proportional, $A = 0$; that if all the elements of a row or column are multiplied by k, the value of the new determinant is kA; that columns and rows may be interchanged, keeping the diagonal elements in their places; and so on.

4. The system of duads (i, k) may be associated with another system of duads (λ, μ), in which λ, μ independently assume the positive integral values 1, 2, 3, etc. The simplest way of doing this is to put

$$\begin{aligned} \lambda &= 2i && \text{if } i \text{ is positive,} \\ &= -2i + 1 && \text{if } i \text{ is zero or negative} \end{aligned}$$

and in like manner for μ with respect to k. This correspondence is reversible; namely, if λ is even, the corresponding value of i is $\frac{1}{2}\lambda$, and if λ is odd, $i = -\frac{1}{2}(\lambda - 1)$; with a similar rule for μ and k.

If, now, we write $a_{ik} = b_{\lambda\mu}$, we can form a new array

$$\begin{matrix} b_{11} & b_{12} & b_{13} \ldots \\ b_{21} & b_{22} & b_{23} \ldots \\ b_{31} & b_{32} & b_{33} \ldots \\ \ldots & \ldots & \ldots \end{matrix}$$

and derive from this a sequence

$$B_1, \; B_2, \ldots B_n, \ldots$$

where $B_n = |b_{nn}|$. This is convergent, and its limit is A; in fact, $B_{2n+1} = A_n$ and B_{2n} is a first minor of A_n which converges to the same limit as A_n itself.

Accordingly it is sufficient to consider infinite determinants associated with an array of the second type indicated. It will be convenient to write

$$|a_{\omega\omega}| = A$$

to express that A is the value of the infinite determinant which is the limit of $|a_{nn}|$ when the positive integer n increases without limit.

5. An infinite determinant is said to be *normal* if the product of the diagonal elements is absolutely convergent, and the sum of all the other elements is absolutely convergent. With the help of Kronecker's symbol δ_{ik}, these conditions are expressed by the single enunciation that the double sum

$$\sum_1^\infty \sum_1^\infty (a_{ik} - \delta_{ik})$$

is absolutely convergent.

Every normal determinant is convergent.

To prove this, we write

$$a_{ik} - \delta_{ik} = b_{ik},$$

and denote the absolute value of b_{ik} by β_{ik}. Putting

$$\bar{P}_n = \prod_{i=1}^{i=n} \{1 + \sum_{k=1}^{k=n} \beta_{ik}\},$$

the product $\overline{P}_n$ is absolutely convergent when n increases without limit, and

$$\operatorname*{Lt}_{n=\infty} (\overline{P}_{n+p} - \overline{P}_n) = 0$$

for all positive integral values of p.

If, now, $$A_n = |a_{nn}|$$

and A_{n+p} is the analogous determinant with p more rows and columns, the difference $(A_{n+p} - A_n)$ can be expanded in terms of the quantities b_{ik}. Changing each term into its absolute value and prefixing the positive sign, we obtain part of the expansion of $(\overline{P}_{n+p} - \overline{P}_n)$. Consequently

$$\text{mod}\,(A_{n+p} - A_n) < \overline{P}_{n+p} - \overline{P}_n,$$

and its limit is zero when $n = \infty$. It follows that $|a_{\omega\omega}|$ is convergent.

6. In a normal determinant A let the element a_{ik} be replaced by unity and all the other elements of the ith row and kth column by zero. The result is a normal determinant which may be denoted by A_{ik} and called a first minor of A. If we put

$$(-1)^{i+k} A_{ik} = \alpha_{ik},$$

we have

$$A = a_{i1}\alpha_{i1} + a_{i2}\alpha_{i2} + \ldots = \sum_{k=1}^{k=\infty} a_{ik}\alpha_{ik},$$

and $$A = a_{1k}\alpha_{1k} + a_{2k}\alpha_{2k} + \ldots = \sum_{i=1}^{i=\infty} a_{ik}\alpha_{ik}.$$

The truth of these formulæ is almost self-evident. Let us take the auxiliary determinant $|a_{nn}|$ where $n > i$; then

$$|a_{nn}| = a_{i1}\bar{\alpha}_{i1} + a_{i2}\bar{\alpha}_{i2} + \ldots + a_{in}\bar{\alpha}_{in},$$

where $\bar{\alpha}_{i1}$, etc. are first minors. Hence

$$\sum_{k=1}^{k=\infty} a_{ik}\alpha_{ik} - |a_{nn}| = \sum_{k=1}^{k=n} a_{ik}(\alpha_{ik} - \bar{\alpha}_{ik}) + \sum_{k=n+1}^{k=\infty} a_{ik}\alpha_{ik}.$$

Now let n increase indefinitely: then the first sum on the right vanishes because each factor $(\alpha_{ik} - \bar{\alpha}_{ik})$ becomes infinitesimal and $\sum_{k=1}^{k=n} a_{ik}$ converges; while the second sum vanishes because

the sequence α_{ik} $(k = n + 1, n + 2, \ldots)$ has an upper limit, and $\sum_{k=n+1}^{k=\infty} a_{ik}$ ultimately vanishes. Therefore

$$\sum_{k=1}^{k=\infty} a_{ik}\alpha_{ik} = \operatorname*{Lt}_{n=\infty} |a_{nn}| = A \,;$$

and the other formula may be proved in a similar manner.

7. The notion of a minor of a normal determinant may be extended as follows. Let

$$p_1, \; p_2, \; \ldots \; p_r$$
$$q_1, \; q_2, \; \ldots \; q_r$$

be any two sets of positive integers, those in each set being all different. In the p_ith row and q_ith column of the normal determinant $|a_{\omega\omega}|$ replace the element $a_{p_i q_i}$ by unity and every other element by zero. After doing this for the values $1, 2, \ldots r$ of i the original determinant has been transformed into another, likewise normal, which we shall denote by

$$\begin{pmatrix} p_1 & p_2 & \cdots & p_r \\ q_1 & q_2 & \cdots & q_r \end{pmatrix}.$$

This is an rth minor of $|a_{\omega\omega}|$ and is complementary to the finite minor

$$|a_{p_i q_j}| \qquad (i, j = 1, 2, \ldots r).$$

If in the determinant $|a_{\omega\omega}|$ we simply omit the rows and columns specified by $p_1 \ldots p_r$ and $q_1 \ldots q_r$ respectively the new determinant is the minor above defined multiplied by the factor $(-1)^{\mu}$, where $\mu = \Sigma\,(p_i + q_i)$.

If $f, g, \ldots l$ is a permutation of $1, 2, \ldots r$ containing λ inversions

$$\begin{pmatrix} p_f & p_g & \cdots & p_l \\ q_f & q_g & \cdots & q_l \end{pmatrix} = (-1)^{\lambda} \begin{pmatrix} p_1 & p_2 & \cdots & p_r \\ q_1 & q_2 & \cdots & q_r \end{pmatrix}.$$

8. From the first two rows of a normal determinant $|a_{\omega\omega}|$ we can derive an aggregate of determinants α_{hk} defined by

$$\alpha_{hk} = \begin{vmatrix} a_{1h} & a_{1k} \\ a_{2h} & a_{2k} \end{vmatrix}$$

where h, k are any two positive integers such that $h < k$. These

finite minors of the second order may be arranged as follows in a linear progression:

$$\alpha_{12}\ \alpha_{13}\ \alpha_{14}\ \alpha_{23}\ \alpha_{15}\ \alpha_{24}\ \alpha_{16}\ldots$$

the rule being that α_{hk} precedes α_{lm} if $h+k<l+m$, or if $h+k=l+m$ and $h<l$. With this notation

$$|a_{\omega\omega}| = \Sigma\alpha_{hk}\begin{pmatrix}1, & 2\\ h, & k\end{pmatrix} \qquad (h, k = 1, 2, 3, \ldots),$$

the terms on the right being arranged according to the rule just explained.

To prove this we take the expansion

$$|a_{nn}| = \Sigma\alpha_{rs}A_{rs} \qquad (r, s = 1, 2, \ldots n)$$
$$= S_1 + S_2,$$

where

$$S_1 = \Sigma\alpha_{hk}A_{hk} \qquad (h+k<n+1),$$
$$S_2 = |a_{nn}| - S_1,$$

and consider what happens when n increases indefinitely. Ultimately the typical term of S_1 converges to

$$\alpha_{hk}\begin{pmatrix}1, & 2\\ h, & k\end{pmatrix},$$

and since all values of h, k are included for which $h+k<n+1$ S_1 itself converges to

$$\Sigma\alpha_{hk}\begin{pmatrix}1, & 2\\ h, & k\end{pmatrix} \qquad (h, k = 1, 2, 3, \ldots)$$

as above defined, provided that, before going to the limit, the terms of S_1 are properly arranged.

The limit of S_2 is zero, because if we write it

$$S_2 = \Sigma\,\alpha_{lm}A_{lm}$$

the absolute values of the quantities A_{lm} have an upper limit, and $\Sigma\alpha_{lm}$ ultimately vanishes. Finally, $|a_{nn}|$ converges to A, the value of $|a_{\omega\omega}|$.

In a similar way, by selecting minors from the first two columns of A, we obtain the expansion

$$|a_{\omega\omega}| = \Sigma\begin{vmatrix}a_{h1} & a_{h2}\\ a_{k1} & a_{k2}\end{vmatrix}\begin{pmatrix}h, & k\\ 1, & 2\end{pmatrix}.$$

9. In a similar way it may be proved that if $p_1, p_2, \dots p_r$ is any fixed selection of r different whole numbers, arranged in ascending order,

$$A = |a_{\omega\omega}| = \sum_q \alpha_{(pq)} \binom{p_1,\ p_2,\ \dots\ p_r}{q_1,\ q_2,\ \dots\ q_r},$$

where on the right hand $\alpha_{(pq)}$ denotes the finite minor

$$\alpha_{(pq)} = |a_{p_i q_j}| \qquad (i, j = 1, 2, \dots r),$$

and $q_1, q_2, \dots q_r$ is any selection of r different whole numbers arranged in ascending order. It must be borne in mind that A is normal, and that the sets $(q_1, q_2, \dots q_r)$ must be arranged in linear order by a suitable rule. The summation includes all the selections $q_1, q_2, \dots q_r$.

There is a corresponding expansion with $q_1, q_2, \dots q_r$ a fixed selection, and $p_1, p_2, \dots p_r$ a variable selection.

10. The product of two normal determinants A, B may be expressed as a normal determinant by a method precisely analogous to that used for finite determinants. Thus, if

$$A = |a_{\omega\omega}|, \quad B = |b_{\omega\omega}|, \quad C = |c_{\omega\omega}|,$$

where

$$c_{ik} = a_{i1}b_{k1} + a_{i2}b_{k2} + \dots = \sum_s a_{is}b_{ks} \qquad (s = 1, 2, 3, \dots),$$

then C is a normal determinant and its value is AB.

To prove that C is normal, write

$$a_{ik} = \delta_{ik} + a'_{ik}, \quad b_{ik} = \delta_{ik} + b'_{ik}, \quad c_{ik} = \delta_{ik} + c'_{ik};$$

then

$$c'_{ik} = a'_{ik} + b'_{ik} + \sum_s a'_{is}b'_{ks},$$

and therefore

$$\Sigma\,|c'_{ik}| \leqslant \Sigma\,|a'_{ik}| + \Sigma\,|b'_{ik}| + \Sigma\,|a'_{is}b'_{ks}| \qquad (i, k, s = 1, 2, 3, \dots).$$

Hence, A, B being normal, $\Sigma\,|c'_{ik}|$ is convergent, and therefore C is normal.

Again, $$AB = \operatorname*{Lt}_{n=\infty} |a_{nn}|\,|b_{nn}| = \mathrm{Lt}\,|\gamma_{nn}|,$$

where

$$\gamma_{ik} = a_{i1}b_{k1} + a_{i2}b_{k2} + \dots + a_{in}b_{kn} \qquad (i, k = 1, 2, 3, \dots n),$$

and it can be proved that $\mathrm{Lt}\,|\gamma_{nn}| = C$.

To do this, write

$$r_{ik} = a_{i,n+1}b_{k,n+1} + a_{i,n+2}b_{k,n+2} + \dots$$
$$= \sum_t a_{it}b_{kt} \qquad (t = n+1, n+2, \dots),$$

then
$$|c_{nn}| = |\gamma_{nn} + r_{nn}|$$
$$= |\gamma_{nn}| + r_{11}C_{11} + r_{12}C_{12} + \dots + r_{1n}C_{1n},$$

where $C_{11}, C_{12}, \dots C_{1n}$ are first minors of $|c_{nn}|$. Now these first minors are all finite determinants, and a positive quantity R can be assigned which is greater than the greatest of their absolute values. Consequently, if ρ_{ik} denotes the absolute value of r_{ik},

$$\text{mod}\,\{|c_{nn}| - |\gamma_{nn}|\} \leqslant R\,(\rho_{11} + \rho_{12} + \dots + \rho_{1n}),$$

which ultimately vanishes when $n = \infty$. Therefore

$$\underset{n=\infty}{\text{Lt}}\,|c_{nn}| = \underset{n=\infty}{\text{Lt}}\,|\gamma_{nn}| = \text{Lt}\,|a_{nn}|\,|b_{nn}| = AB,$$

which proves the theorem.

As in the case of finite determinants the product AB may be constructed in four different ways (v. 4).

11. There is a class of infinite determinants, which we shall call *semi-normal*, defined as follows. Let

$$\begin{cases} x_1,\ x_2,\ x_3,\ \dots \\ y_1,\ y_2,\ y_3,\ \dots \end{cases}$$

be a series of quantities such that

$$P_\omega = \prod_1^\infty \frac{x_r}{y_r}$$

is absolutely convergent. Suppose also that

$$A = |a_{\omega\omega}|, \quad B = |b_{\omega\omega}|,$$

where
$$b_{ik} = \frac{x_i}{y_k}\,a_{ik}.$$

Then if A is normal, B is also normal: but it may happen that B is normal when A is not. In this case A is said to be semi-normal, and the system (x_r, y_r) may be called a reducent of A. Clearly if one reducent exists, there will be any number of them.

Since
$$|b_{nn}| = P_n\,|a_{nn}|,$$

where P_n is the product of the first n factors of P_ω, and since

when n is infinite $|b_{nn}|$ and P_n converge to the limits B and P_ω it follows that A is convergent, and that

$$A = P_\omega^{-1}B.$$

12. Under certain conditions the product of two semi-normals may be expressed as a semi-normal.

Let $$A = |a_{\omega\omega}|, \quad B = |b_{\omega\omega}|$$

be two semi-normals with the respective reducents

$$\begin{cases} x_1 & x_2 & x_3 & \dots \\ y_1 & y_2 & y_3 & \dots \end{cases} \quad \begin{cases} z_1 & z_2 & z_3 & \dots \\ u_1 & u_2 & u_3 & \dots \end{cases}$$

then if the product

$$P_\omega = \prod_1^\infty \frac{x_r}{u_r}$$

is absolutely convergent,

$$AB = C = |c_{\omega\omega}|,$$

where $$c_{ik} = \Sigma a_{ih} b_{hk} \qquad (h = 1, 2, 3, \dots),$$

and C is a semi-normal determinant of which

$$\begin{cases} x_1 & x_2 & x_3 & \dots \\ u_1 & u_2 & u_3 & \dots \end{cases}$$

is a reducent.

To prove this, let us write

$$\alpha_{ik} = \frac{x_i}{y_k} a_{ik}, \quad \beta_{ik} = \frac{z_i}{u_k} b_{ik}, \quad \gamma_{ik} = \sum_1^\infty \alpha_{ih}\beta_{hk};$$

then $$c_{ik} = \Sigma \frac{y_h}{x_i} \alpha_{ih} \frac{u_k}{z_h} \beta_{hk} = \frac{u_k}{x_i} \Sigma \frac{y_h}{z_h} \alpha_{ih}\beta_{hk},$$

and therefore $$\text{mod } c_{ik} \leqslant \mu \text{ mod } \left(\frac{u_k}{x_i}\gamma_{ik}\right),$$

where μ is the upper limit of the quantities mod (y_h/z_h). This upper limit exists, and is finite, because

$$\prod_1^\infty \frac{y_h}{z_h} = \Pi \frac{y_h}{x_h} \cdot \Pi \frac{x_h}{u_h} \cdot \Pi \frac{u_h}{z_h},$$

all the products on the right-hand being absolutely convergent. Thus the series c_{ik} is absolutely convergent.

Again if we put

$$\alpha_{ik} = \delta_{ik} + \alpha'_{ik}, \quad \beta_{ik} = \delta_{ik} + \beta'_{ik},$$

where δ_{ik}, as usual, is Kronecker's symbol,

$$c_{ii} = \frac{u_i y_i}{x_i z_i}\left(1 + \alpha'_{ii} + \beta'_{ii} + \sum_h \frac{z_i}{y_i}\frac{y_h}{z_h}\alpha'_{ih}\beta'_{hi}\right),$$

and hence the product Πc_{ii} is absolutely convergent, and so also is $\Pi \frac{x_i}{u_i} c_{ii}$.

Finally the series

$$\sum_{i,k} \frac{x_i}{u_k} c_{ik} = \Sigma \frac{y_h}{z_h} \alpha_{ih}\beta_{hk}$$

is absolutely convergent, because $\Sigma\alpha_{ih}\beta_{hk}$ is so and y_h/z_h has a finite upper limit. The theorem stated has therefore been proved.

In the same way it can be shewn that if $\Pi x_h z_h$ is absolutely convergent

$$|a_{\omega\omega}|\,|b_{\omega\omega}| = |d_{\omega\omega}|,$$

where $|d_{\omega\omega}|$ is a semi-normal with elements defined by

$$d_{ik} = \sum_h a_{ih} b_{kh},$$

and a reducent

$$\begin{cases} x_1, & x_2, \quad x_3 \quad \ldots \\ u_1^{-1}, & u_2^{-1}, \quad u_3^{-1} \ldots \end{cases}.$$

Moreover, in these enunciations, x and y can be interchanged, and also z and u.

CHAPTER XI.

APPLICATIONS TO THE THEORY OF EQUATIONS AND OF ELIMINATION.

1. THE solution of a system of linear equations has already been partially considered (p. 26); we shall now proceed to discuss the general problem. Let us take the m homogeneous equations

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = 0 \end{array}\right\};$$

the question is to find all the values of the unknown quantities x_i which satisfy these relations.

The nature of the solution is essentially connected with the matrix (a_{mn}), which we shall denote by A. It follows from the partial investigation above referred to that if $n = m + 1$ and A is of rank m, the ratios of $x_1, x_2, \ldots x_n$ are determinate: in fact

$$x_1 : x_2 : \ldots : x_n = A_1 : A_2 : \ldots : A_n,$$

where $A_1, A_2, \ldots A_n$ are determinants of the nth order derived from A by suppressing one column. If, however, the rank of A is less than m, the determinants A_i all vanish, and the values of $x_1 : x_2 : \ldots : x_n$ apparently become indeterminate: the process of p. 26 is in fact illegitimate, because the derived system is not equivalent to the given one. We shall see presently that if r is the rank of A, and $m < n$, the complete solution will involve $n - r$ independent parameters, while if $m \geqslant n$, the only solution is $x_1 = x_2 = \ldots = x_n = 0$ unless $r < n$, in which case there is a solution involving $n - r$ parameters.

2. Suppose, in the first place, that $m = n$, and that $|a_{nn}|$ is not zero. Let the reciprocal of $|a_{nn}|$ be $|\alpha_{nn}|$, and write

$$u_i = \Sigma a_{ik} x_k, \qquad (k = 1, 2, \dots n).$$

Then we have identically

$$\sum_k \alpha_{ik} u_i = \sum_{i,k} a_{is} \alpha_{ik} x_s = |a_{nn}|\, x_i,$$

and hence

$$|a_{nn}|\, x_i = 0,$$

or, since $|a_{nn}|$ does not vanish,

$$x_1 = x_2 = \dots = x_n = 0.$$

This, then, is the only solution when $|a_{nn}|$ is different from zero.

Conversely if the equations $u_i = 0$ can be satisfied by values of $x_1, x_2, \dots x_n$ which are not all zero, the determinant $|a_{nn}|$ must vanish. This determinant is called the resultant (or eliminant) of the n equations $u_i = 0$: or again the determinant of the n linear forms u_i.

3. Next suppose that $m < n$. Without loss of generality we may assume that $|a_{rr}|$ does not vanish: let this determinant be called A_r and let its reciprocal be $|\alpha_{rr}|$. Then as in last article

$$\alpha_{1i} u_1 + \alpha_{2i} u_2 + \dots + \alpha_{ri} u_r = A_r x_i + \sum_{s=r+1}^{s=n} C_{i,s} x_s, \quad (i = 1, 2, \dots r),$$

where $C_{i,s}$ is a minor of A of order r, which may or may not vanish. Hence instead of the system $u_1 = u_2 = \dots = u_r = 0$ we have the derived system

$$A_r x_i = -\sum_{r+1}^{n} C_{is} x_s, \qquad (i = 1, 2, \dots r),$$

which is equivalent to it because $|\alpha_{rr}| = A_r^{r-1}$, which is different from zero.

Hence we obtain $x_1, x_2, \dots x_r$ as definite linear functions of the $(n-r)$ quantities $x_{r+1}, x_{r+2}, \dots x_n$: it may of course happen that some, or even all, of the coefficients C_{is} vanish. In any case, we have found a complete solution of

$$u_1 = 0, \;\; u_2 = 0, \dots \; u_r = 0,$$

with $(n-r)$ independent parameters $x_{r+1}, \dots x_n$. Now let A_{r+1}

be the vanishing determinant $|a_{r+1,r+1}|$ and $|\beta_{r+1,r+1}|$ its reciprocal in which $\beta_{r+1,r+1} = A_r$, a quantity different from zero. Then we have an identity

$$\beta_{1,r+1} u_1 + \beta_{2,r+1} u_2 + \ldots + \beta_{r+1,r+1} u_{r+1} = 0,$$

because if the expression on the left is arranged as a linear function of $x_1, x_2, \ldots x_n$ every coefficient vanishes as being a minor of A which is of order $(r+1)$. Hence every solution of the system $u_1 = u_2 = \ldots = u_r = 0$ will also satisfy $u_{r+1} = 0$: and in the same way it will satisfy $u_s = 0$ for $s = r+2, r+3, \ldots m$. Thus we have actually obtained the most general solution of the given system of m equations in a form which leaves $(n-r)$ of the unknown quantities arbitrary. It may happen that some one or more of the quantities $x_1, \ldots x_r$ is definitely zero. For instance, the solution of

$$x + y - 2z = 0,$$
$$2x + y - 2z = 0,$$

is $x = 0$, $y = 2\lambda$, $z = \lambda$, with λ arbitrary.

4. Next let $m > n$, so that the given system is redundant. Then if $r = n$ the only solution is $x_i = 0$, because without loss of generality we may suppose that $|a_{nn}|$ is different from zero, and then the first n equations give $x_i = 0$. If $r < n$, we may suppose that $|a_{rr}|$ is not zero, and proceed as in last article: we thus again find a general solution in which $(n-r)$ of the unknown quantities remain arbitrary.

5. Having given m linear homogeneous functions u_i of n independent variables $x_1, x_2, \ldots x_n$, it is important to know how many independent identities exist of the form

$$L(u) = \lambda_1 u_1 + \lambda_2 u_2 + \ldots + \lambda_m u_m = 0,$$

with constant coefficients λ_i.

Equating to zero the coefficients of $x_1, x_2, \ldots x_n$ on the left hand we have

$$\begin{gathered}
a_{11}\lambda_1 + a_{21}\lambda_2 + \ldots + a_{m1}\lambda_m = 0, \\
a_{12}\lambda_1 + a_{22}\lambda_2 + \ldots + a_{m2}\lambda_m = 0, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
a_{1n}\lambda_1 + a_{2n}\lambda_2 + \ldots + a_{mn}\lambda_m = 0.
\end{gathered}$$

This is a system with the matrix $(a_{mn})'$, the rank of which is the same as that of (a_{mn}). Consequently if r is not less than the smaller of the numbers m, n, the system will have only a zero solution and the forms u_i will be independent: while if r is below this limit there will be $(m-r)$ linear relations $L_i(u)=0$ from which all others can be derived in the form

$$\Sigma C_i L_i(u)=0, \qquad (i=1,\ 2,\ \ldots\ \overline{m-r}),$$

with arbitrary coefficients C_i.

6. A non-homogeneous system

$$u_i+c_i=0, \qquad (i=1,\ 2,\ \ldots\ m),$$

where c_i is a constant, may be reduced to a homogeneous one by putting

$$x_k=\frac{y_k}{y_{n+1}}, \qquad (k=1,\ 2,\ \ldots\ n);$$

it should be noticed that for particular values of the arbitrary parameters contained in the solution x_i may become infinite or indeterminate.

7. Suppose that we have two sets of variables $x_1,\ x_2,\ \ldots\ x_n$ and $y_1,\ y_2,\ \ldots\ y_n$ connected by the relations

$$y_i=\sum_k a_{ik}x_k, \qquad (i,\ k=1,\ 2,\ \ldots\ n),$$

the coefficients a_{ik} being constant, and $|a_{nn}|$ different from zero. From these equations we can deduce an equivalent set

$$x_i=\sum_k \bar{a}_{ik}y_k.$$

If we substitute in the first set of equations the value of x_i given by the second set we obtain n linear equations in $y_1,\ y_2, \ldots\ y_n$ which must be identities, if, as we suppose, the variables

$$x_1,\ x_2,\ \ldots\ x_n$$

are independent. Hence

$$\sum_k a_{ik}\bar{a}_{kj}=\delta_{ij},$$

and in the same way by substituting from the first set of equations in the second

$$\Sigma\bar{a}_{ik}a_{kj}=\delta_{ij}.$$

Consequently

$$(a_{nn})(\bar{a}_{nn}) = (\bar{a}_{nn})(a_{nn}) = [1], \qquad |a_{nn}||\bar{a}_{nn}| = 1.$$

If we denote the matrix (a_{nn}) by A, and write E for the matrix [1], then the matrix $(\bar{a}_{nn})$ is conveniently denoted by A^{-1}: thus

$$AA^{-1} = A^{-1}A = E.$$

This notation is consistent with the ordinary laws of indices; and we may express the relation between the sets $(x_1, x_2, \dots x_n)$ and $(y_1, y_2, \dots y_n)$ in either of the symbolical forms

$$(y) = A(x), \quad (x) = A^{-1}(y).$$

These relations constitute what is called a linear substitution; thus in analytical geometry when we change from one set of coordinates to another of the same type, this is effected by means of a linear transformation.

8. If there are three sets of n variables (x), (y), (z) such that

$$(y) = A(x), \quad (z) = B(y),$$

then it is found by direct elimination of $y_1, y_2, \dots y_n$ that

$$(z) = BA(x), \quad x = (BA)^{-1}(z) = A^{-1}B^{-1}(z),$$

the products BA, $A^{-1}B^{-1}$ being defined as in v. 2 (p. 50). This theorem may obviously be generalised.

Let $u(x)$ be the linear form defined by

$$u(x) = \Sigma \xi_k x_k, \qquad (k = 1, 2, \dots n),$$

then by the substitution $(x) = A(y)$ this is converted into

$$v(y) = \Sigma \eta_k y_k,$$

where $$\eta_k = \xi_1 a_{1k} + \xi_2 a_{2k} + \dots + \xi_n a_{nk},$$

so that $$(\eta) = A'(\xi),$$

where A' is the conjugate of A (p. 49).

Thus the simultaneous substitutions $(x) = A(y)$, $(\eta) = A'(\xi)$ transform $\Sigma \xi_i x_i$ into $\Sigma \eta_i y_i$. The variables ξ_i are said to be contragredient to the variables x_i. Variables transformed by the same substitution are said to be cogredient.

9. Let there be n linear forms $u_i(x)$ defined by

$$u_i(x) = \sum_k p_{ik} x_k,$$

and let them be transformed into linear forms $v_i(y)$ by the substitution $x = A(y)$. Then

$$v_i(y) = \sum q_{ik} y_k,$$

where

$$q_{ik} = \sum_r p_{ir} a_{rk},$$

and consequently

$$|q_{nn}| = |p_{nn}| \cdot |a_{nn}|.$$

Thus the determinant of the system of forms reproduces itself multiplied by $|a_{nn}|$, which is called the modulus of the transformation. This is what might have been expected: for if $|p_{nn}| = 0$ the quantities $u_i(x)$ are not independent: and when this is so, the quantities $v_i(y)$ are not independent either, so that $|q_{nn}| = 0$.

Given any system of forms $F_i(x_1, x_2, \ldots x_n)$ the substitution $(x) = A(y)$ converts them into forms $G_i(y_1, y_2, \ldots y_n)$. If a function of the new coefficients is identically equal to the same function of the old coefficients multiplied by a power of the modulus of transformation, we have what is called an invariant of the system of forms. It has been proved, then, that the eliminant of a system of n linear forms in n variables is an invariant.

It follows from Arts. 1—5 of this chapter, as well as from VII. 8, 9, that the rank of $|q_{nn}|$ is the same as that of $|p_{nn}|$.

10. In Art. 2 we have the first example of the process of elimination; namely, we have found a condition, independent of the variables, which must hold if a certain given number of equations are to exist between these variables. When r homogeneous equations hold between r variable quantities, (or what is the same thing, r non-homogeneous equations between $r-1$ quantities), it is always possible to establish an equation $R = 0$ between the coefficients of these equations alone. Then R is called the resultant or eliminant of the system of equations.

When the equations are two in number the most direct process is Sylvester's dialytic method. Let the two equations be

$$\left.\begin{aligned}0 &= a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m \\ 0 &= b_0 + b_1 x + b_2 x^2 + \ldots + b_n x^n\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots(1).$$

If we multiply the first equation by $1, x, x^2 \ldots x^{n-1}$ we get $n-1$ new equations, and from the second by multiplying by $1, x, x^2 \ldots x^{m-1}$ we get $m-1$ new equations, viz. we have now the system

$$\begin{aligned}
0 &= a_0 + a_1 x + a_2 x^2 + \ldots \\
0 &= \phantom{a_0 + {}} a_0 x + a_1 x^2 + \ldots \\
0 &= \phantom{a_0 + a_1 x + {}} a_0 x^2 + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
0 &= b_0 + b_1 x + b_2 x^2 + \ldots \\
0 &= \phantom{b_0 + {}} b_0 x + b_1 x^2 + \ldots \\
0 &= \phantom{b_0 + b_1 x + {}} b_0 x^2 + \ldots \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

of $m+n$ equations satisfied by the same values of x as the given equations (1) and linear and homogeneous in the $m+n$ quantities

$$1,\ x,\ x^2 \ldots x^{m+n-1}.$$

Hence, by Art. 3, the determinant of the system must vanish, or

$$R = \begin{vmatrix} a_0, & a_1, & a_2 & \ldots\ldots \\ & a_0, & a_1 & \ldots\ldots \\ & & a_0 & \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \\ b_0, & b_1, & b_2 & \ldots\ldots \\ & b_0, & b_1 & \ldots\ldots \\ & & b_0 & \ldots\ldots \end{vmatrix} = 0,$$

the determinant being of order $m+n$. Since there are n rows of a's, and m of b's, the resultant is of order n in the coefficients of the first equation, and of order m in the coefficients of the second.

11. If the coefficients $a_m,\ a_{m-1},\ a_{m-2} \ldots b_n,\ b_{n-1},\ b_{n-2} \ldots$ are functions of y and z of degrees $0, 1, 2 \ldots$, it can be proved that the resultant is of order mn in y and z. This will be the case if every term in R has the sum of the complements of the suffixes equal to mn.

If we change y and z into yt and zt respectively, the value of R is now

$$R' = \begin{vmatrix} a_0 t^m, & a_1 t^{m-1}, & a_2 t^{m-2} & \dots \\ & a_0 t^m, & a_1 t^{m-1} & \dots \\ \dots & \dots & \dots & \dots \\ b_0 t^n, & b_1 t^{n-1}, & b_2 t^{n-2} & \dots \\ & b_0 t^n, & b_1 t^{n-1} & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix}.$$

Observe that the separate elements and therefore each term of R' is multiplied by a power of t equal to the complement of the suffix.

Now, multiply the first n rows by

$$t^{n-1},\ t^{n-2} \dots t,\ 1,$$

and the last m by

$$t^{m-1},\ t^{m-2}, \dots t,\ 1.$$

Then R' is multiplied by a power of t, whose exponent is

$$\frac{m(m-1)}{2} + \frac{n(n-1)}{2}.$$

But now the first column of R' divides by t^{m+n-1}, the second by t^{m+n-2}, and so on. Thus $R' \div R$ is equal to a power of t whose exponent is

$$\frac{(m+n)(m+n-1)}{2} - \frac{m(m-1)}{2} - \frac{n(n-1)}{2} = mn.$$

Thus every term in R' must divide by t^{mn}, which proves the theorem. Functions, such that the sum of the suffixes, or of their complements, of the elements in each term is constant, are sometimes called *isobaric*, and the constant sum is called the weight.

12. We may consider the question in another way.

If
$$\begin{aligned} \phi(x) &= b_0 + b_1 x + b_2 x^2 + \dots + b_n x^n \\ &= b_n (x - \beta_1)(x - \beta_2) \dots (x - \beta_n) \quad \dots\dots\dots\dots\dots (1) \end{aligned}$$

is an equation whose roots are $\beta_1,\ \beta_2 \dots \beta_n$, the function

$$f(x) = u = a_0 + a_1 x + a_2 x^2 + \dots + a_m x^m \quad \dots\dots\dots\dots (2)$$

has n values corresponding to the different values of x given by (1). These n values are the roots of an equation of the nth degree, which we now proceed to find. Multiply the equations (1) and (2)

by the same powers of x as in Art. 9, and we have the $m+n$ equations

$$\begin{array}{l} 0 = a_0 - u + a_1 x + a_2 x^2 + \dots \\ 0 = \quad (a_0 - u)\, x + a_1 x^2 + \dots \\ 0 = \qquad (a_0 - u)\, x^2 + \dots \\ \quad \dots\dots\dots\dots\dots\dots\dots\dots \quad . \\ 0 = \quad b_0 + b_1 x + b_2 x^2 + \dots \\ 0 = \qquad b_0 x + b_1 x^2 + \dots \\ 0 = \quad \dots\dots\dots\dots\dots\dots\dots \end{array}$$

Eliminating between these the quantities

$$x^{m+n-1} \dots x, \ 1,$$

we get

$$\begin{vmatrix} a_0 - u, & a_1, & a_2 & \dots \\ & a_0 - u, & a_1 & \dots \\ & & a_0 - u & \dots \\ & \dots\dots & \dots\dots & \dots \\ b_0, & b_1, & b_2 & \dots \\ & b_0, & b_1 & \dots \\ & \dots\dots & \dots\dots & \dots \end{vmatrix} = 0,$$

an equation of the nth degree to find u, the roots of which are

$$f(\beta_1), \ f(\beta_2), \dots f(\beta_n).$$

The product of the roots being equal to the constant term,

$$(-1)^n b_n^m f(\beta_1) f(\beta_2) \dots f(\beta_n) = (-1)^n R,$$

where R has the meaning in Art. 10. Thus

$$R = b_n^m f(\beta_1) f(\beta_2) \dots f(\beta_n).$$

In the same way we may shew that

$$R = (-1)^{mn} (a_m^n)\, \phi(\alpha_1)\, \phi(\alpha_2) \dots \phi(\alpha_m)$$

if $\alpha_1 \dots \alpha_m$ are the roots of (2).

This result shews that the value of R obtained in Art. 10 does not involve any irrelevant factor; for clearly

$$\phi(\alpha_1)\, \phi(\alpha_2) \dots \phi(\alpha_m)$$

is the simplest rational symmetric function of both sets of roots which vanishes when the equations have a root in common.

13. If the two functions ϕ and f of the preceding article are a function and its differential coefficient, then R is called the discriminant of the function, and its vanishing is the condition that the function should have equal roots. If

$$
\begin{aligned}
f(x) &= a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \\
&= a_n (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n) \\
f'(x) &= \qquad a_1 + 2a_2 x + \dots + n a_n x^{n-1}, \\
R &= a_n^{n-1} f'(\alpha_1) f'(\alpha_2) \dots f'(\alpha_n) \\
&= \begin{vmatrix} a_1, & 2a_2, & 3a_3 \dots \\ & a_1, & 2a_2 \dots \\ \dots\dots & \dots\dots & \dots\dots \\ a_0, & a_1, & a_2 \dots \\ & a_0, & a_1 \dots \\ \dots\dots & \dots\dots & \dots\dots \end{vmatrix},
\end{aligned}
$$

having n rows of the first, and $n-1$ of the second kind.

If we multiply the last row by n, and subtract it from the nth, this becomes

$$0 \dots 0, \; -na_0, \; -(n-1)a_1, \dots -a_{n-1}, \; 0.$$

Thus the determinant reduces into the product of a_n by a determinant of order $2n-2$, which we shall call Δ.

Also

$$
\begin{aligned}
f'(\alpha_1) &= \quad a_n \quad (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3) \dots (\alpha_1 - \alpha_n) \\
f'(\alpha_2) &= (\alpha_2 - \alpha_1) \quad a_n \quad (\alpha_2 - \alpha_3) \dots (\alpha_2 - \alpha_n) \\
&\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\
f'(\alpha_n) &= (\alpha_n - \alpha_1)(\alpha_n - \alpha_2)(\alpha_n - \alpha_3) \dots \quad a_n;
\end{aligned}
$$

$$\therefore f'(\alpha_1) f'(\alpha_2) \dots f'(\alpha_n) = (-1)^{\frac{n(n-1)}{2}} a_n^n \zeta(\alpha_1, \alpha_2 \dots \alpha_n)$$

where $\zeta(\alpha_1 \dots \alpha_n)$ means the product of the squares of the differences of all the roots. Thus

$$\Delta = (-1)^{\frac{n(n-1)}{2}} a_n^{2n-2} \zeta(\alpha_1, \alpha_2 \dots \alpha_n).$$

14. The artifice used in eliminating x between two equations may sometimes be employed for the case of more equations than two, as in the following examples due to Cayley.

Let $\qquad x + y + z = 0, \quad x^2 = a, \quad y^2 = b, \quad z^2 = c;$

multiply the first equation by 1, yz, zx, xy, and reduce by means of the other three, then we get

$$\begin{aligned} x + y \quad + \quad z &= 0 \\ xyz \qquad + cy + bz &= 0 \\ xyz + cx \qquad + az &= 0 \\ xyz + bx + ay \qquad &= 0, \end{aligned}$$

whence, eliminating xyz, x, y, z, we get

$$\begin{vmatrix} ., & 1, & 1, & 1 \\ 1, & ., & c, & b \\ 1, & c, & ., & a \\ 1, & b, & a, & . \end{vmatrix} = 0.$$

Or if we multiply the equation by x, y, z, xyz, and eliminate 1, yz, zx, xy, we get

$$\begin{vmatrix} ., & a, & b, & c \\ a, & ., & 1, & 1 \\ b, & 1, & ., & 1 \\ c, & 1, & 1, & . \end{vmatrix} = 0.$$

Again, if we are given the equations

$$x + y + z = 0, \quad x^3 = a, \quad y^3 = b, \quad z^3 = c,$$

if we multiply the first equation by

$$x, \ y, \ z, \ y^2z^2, \ z^2x^2, \ x^2y^2, \ x^2yz, \ y^2zx, \ z^2xy,$$

and reduce by the last three we can eliminate

$$x^2, \ y^2, \ z^2, \ yz, \ zx, \ xy, \ xy^2z^2, \ yz^2x^2, \ zx^2y^2$$

between the resulting equations, giving

$$\begin{vmatrix} 1, & ., & ., & ., & 1, & 1, & ., & ., & . \\ ., & 1, & ., & 1, & ., & 1, & ., & ., & . \\ ., & ., & 1, & 1, & 1, & ., & ., & ., & . \\ ., & c, & b, & ., & ., & ., & 1, & ., & . \\ c, & ., & a, & ., & ., & ., & ., & 1, & . \\ b, & a, & ., & ., & ., & ., & ., & ., & 1 \\ ., & ., & ., & a, & ., & ., & ., & 1, & 1 \\ ., & ., & ., & ., & b, & ., & 1, & ., & 1 \\ ., & ., & ., & ., & ., & c, & 1, & 1, & . \end{vmatrix} = 0.$$

Other forms of the resultant can also be obtained.

15. The resultant of two equations has been obtained in a compact form by Bézout. Let the equations be

$$f = a_0x^m + a_1x^{m-1} + \dots + a_m = 0,$$
$$\phi = b_0x^n + b_1x^{n-1} + \dots + b_n = 0,$$

and suppose $m \geqslant n$.

Write

$$f_0 = a_0,\ f_1 = a_0x + a_1,\ f_2 = a_0x^2 + a_1x + a_2, \dots f_r = a_0x^r + a_1x^{r-1} + \dots + a_r$$
$$\phi_0 = b_0,\ \phi_1 = b_0x + b_1,\ \phi_2 = b_0x^2 + b_1x + b_2, \dots \phi_r = b_0x^r + b_1x^{r-1} + \dots + b_r$$

and form the combinations

$$\begin{aligned} X_r &= x^{m-n} f_r \phi - \phi_r f \\ &= x^{m-n} f_r (b_{r+1}x^{n-r-1} + \dots + b_n) - \phi_r (a_{r+1}x^{m-r-1} + \dots + a_m) \\ &= \sum_s c_{rs}x^{s-1}, \qquad (s = 1, 2, 3, \dots m), \end{aligned}$$

for $r = 0, 1, 2 \dots (n-1)$. If $f = 0$ and $\phi = 0$ have a common root the m equations

$$X_0 = 0,\ X_1 = 0, \dots X_{n-1} = 0,\ \phi = 0,\ x\phi = 0, \dots x^{m-n-1}\phi = 0,$$

can be simultaneously satisfied, and hence eliminating

$$1,\ x,\ x^2, \dots x^{m-1}$$

dialytically

$$\begin{vmatrix} c_{0,1}, & c_{0,2}, & \dots & c_{0,m} \\ \dots & \dots & \dots & \dots \\ c_{n-1,1}, & c_{n-1,2}, & \dots & c_{n-1,m} \\ b_0, & b_1, & \dots & 0 \\ 0, & b_0, & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & \dots & b_n \end{vmatrix} = 0.$$

It is easily seen that the expression on the left is of the proper dimensions in the coefficients of f and ϕ, and that it does not vanish identically: hence it is the resultant, free from extraneous factors.

As an example, let $m = 3$, $n = 2$: then the resultant is

$$\begin{vmatrix} a_0b_1 - a_1b_0, & a_0b_2 - a_2b_0, & -a_3b_0 \\ a_0b_2 - a_2b_0, & a_1b_2 - a_2b_1, & -a_3b_1 \\ b_0, & b_1, & b_2 \end{vmatrix}.$$

When $m = n$, the determinant is symmetrical, and each of its elements is expressible as a sum of quantities of the type

$$(a_i b_j - a_j b_i).$$

If in this symmetrical determinant of order m, we put

$$b_{n+1} = b_{n+2} = \dots = b_m = 0,$$

the resulting expression is the product of a_m^{m-n} by the resultant of f and ϕ as they are given at the beginning of this article.

16. We shall now prove that if Bézout's determinant is of rank r, the polynomials f, ϕ have a highest common factor H of degree $(m - r)$. For the sake of a uniform notation, let

$$\psi_1 = \phi,\ \psi_2 = x\phi,\ \dots\ \psi_{m-n} = x^{m-n}\phi,$$
$$\psi_{m-n+1} = X_0,\ \psi_{m-n+2} = X_1,\ \dots\ \psi_m = X_{n-1},$$
$$\psi_i = \Sigma p_{ik} x^{k-1}, \qquad (k = 1, 2, \dots m),$$

so that the resultant is

$$R_m = |p_{mm}|.$$

Let $\lambda_1, \lambda_2, \dots \lambda_m$ be constants, t any whole number not greater than m: then if we write

$$\Psi = \lambda_1\psi_1 + \lambda_2\psi_2 + \dots + \lambda_m\psi_m = C_0 + C_1 x + \dots + C_{m-1}x^{m-1},$$

the conditions

$$C_{m-1} = C_{m-2} = \dots = C_{m-t} = 0$$

form a deficient system of linear equations in $\lambda_1, \lambda_2, \dots \lambda_m$. This system always has a solution in which the quantities λ_i are not all zero (Art. 3). For such values the degree of Ψ is less than $m - t$. Now Ψ can be expressed in the form $Af - B\phi$ where A, B are polynomials, so that Ψ is divisible by H. The degree of H cannot exceed m: let it be $m - \mu$. Then by putting $t = \mu$, we infer that from the equations

$$C_{m-1} = C_{m-2} = \dots = C_{m-\mu} = 0$$

the other equations $C_0 = C_1 = \dots = C_{m-\mu-1} = 0$ necessarily follow: in other words the system

$$C_0 = C_1 = \dots = C_{m-1} = 0$$

does not contain more than μ independent equations. But the actual number of independent relations is precisely r: consequently

$$\mu \geqslant r,$$

and the degree of H cannot exceed $m - r$.

The general values $(\lambda_1, \lambda_2, \ldots \lambda_m)$ which make Ψ identically zero can be expressed as linear homogeneous functions of $(m-r)$ arbitrary parameters. Now in the identity $\Psi = Af - B\phi$ the value of B is

$$-(\lambda_1 + \lambda_2 x + \ldots + \lambda_{m-n} x^{m-n}) \\ - x^{m-n}(\lambda_{m-n+1} f_0 + \lambda_{m-n+2} f_1 + \ldots + \lambda_m f_{n-1})$$

of which the degree is $m-1$. Equating to zero the coefficients of

$$x^{m-1},\ x^{m-2},\ \ldots\ x^{r+2},\ x^{r+1},$$

we get a system of $(m-r-1)$ linear equations which determine the ratios of the $(m-r)$ parameters: substituting these in A and B we get an identity

$$A'f - B'\phi = 0,$$

in which the degree of B' does not exceed r. Since $B'\phi$ is divisible by f, it follows that f, ϕ have a common divisor of degree not less than $m-r$. We have already seen that the degree of H cannot exceed $m-r$: therefore its degree is exactly $m-r$, as stated. In fact $H = f/B' = \phi/A'$.

A numerical example will illustrate the argument. Let

$$f = x^4 + x^3 + 2x^2 + x + 1,$$
$$\phi = x^3 + x^2 + x + 1:$$

then

$$R = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 \\ 0 & -1 & 0 & -1 \end{vmatrix},$$

$$\psi_1 = \phi = x^3 + x^2 + x + 1, \quad \psi_3 = (x^2 + x)\phi - (x+1)f = -x^3 - x^2 - x - 1,$$
$$\psi_2 = x\phi - f = -x^2 - 1, \quad \psi_4 = (x^3 + x^2 + 2x)\phi - (x^2 + x + 1)f = -x^2 - 1.$$

The identity $\lambda_1\psi_1 + \lambda_2\psi_2 + \lambda_3\psi_3 + \lambda_4\psi_4 = 0$

requires that

$$\lambda_1 - \lambda_3 = 0,$$
$$\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 = 0,$$
$$\lambda_1 - \lambda_3 = 0,$$
$$\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4 = 0,$$

whence

$$\lambda_1, \lambda_2, \lambda_3, \lambda_4 = t,\ u,\ t,\ -u,$$

where t, u are arbitrary. This gives

$$0 = \Sigma \lambda_i \psi_i = [t(x^2+x+1) - u(x^3+x^2+x)]\,\phi$$
$$- [t(x+1) - u(x^2+x)]\,f.$$

To reduce the degree of the polynomial which multiplies ϕ, we must put $u = 0$: thus finally,

$$(x^2+x+1)\,\phi - (x+1)f = 0$$

identically, and $H = \phi/(x+1) = x^2+1$.

17. The resultant of the quadric

$$u = a_{11}x_1^2 + \ldots + 2a_{ik}x_ix_k + \ldots = 0 \quad \ldots\ldots\ldots\ldots(1),$$

and of the $n-1$ linear equations

$$\left.\begin{array}{l} v_1 = c_{11}x_1 + c_{12}x_2 + \ldots + c_{1n}x_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ v_{n-1} = c_{n-11}x_1 + c_{n-12}x_2 + \ldots + c_{n-1n}x_n = 0 \end{array}\right\} \ldots\ldots\ldots(2)$$

can be readily expressed as a determinant.

By Euler's theorem for homogeneous functions we can write the first equation in the form

$$x_1\frac{du}{dx_1} + x_2\frac{du}{dx_2} + \ldots + x_n\frac{du}{dx_n} = 2u = 0 \quad \ldots\ldots\ldots\ldots(3).$$

Then if in equation (3) we do not consider the variables implicitly contained in the differential coefficients, (1) and (2) being n equations between $x_1 \ldots x_n$, (3) must be capable of being put in the form

$$\lambda_1 v_1 + \lambda_2 v_2 + \ldots + \lambda_{n-1}v_{n-1} = 0 \quad \ldots\ldots\ldots\ldots\ldots(4).$$

Equating coefficients in (3) and (4),

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = \lambda_1 c_{11} + \lambda_2 c_{21} + \ldots + \lambda_{n-1}c_{n-11} \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = \lambda_1 c_{12} + \lambda_2 c_{22} + \ldots + \lambda_{n-1}c_{n-12} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = \lambda_1 c_{1n} + \lambda_2 c_{2n} + \ldots + \lambda_{n-1}c_{n-1n} \end{array}\right\} \ (5).$$

The equations (5) together with (2) form a system of $2n-1$ equations between $x_1, x_2 \ldots x_n, \lambda_1, \lambda_2 \ldots \lambda_{n-1}$; hence their determinant must vanish. Thus

$$\begin{vmatrix} a_{11} & \dots a_{1n}, & c_{11} & \dots c_{n-11} \\ \dots & \dots & \dots & \dots \\ a_{n1} & \dots a_{nn}, & c_{1n} & \dots c_{n-1n} \\ c_{11} & \dots c_{1n} & & \\ \dots & \dots & & \\ c_{n-11} & \dots c_{n-1n} & & \end{vmatrix} = 0,$$

the blank space being filled with zeros. This result is due to Versluijs. a_{ik} and a_{ki} mean the same thing, viz. half the coefficient of $x_i x_k$ in the quadric.

18. The system of equations

$$x + y = a, \quad x^2 + y^2 = b^2,$$

is solved by establishing the new linear equation

$$x - y = \pm \sqrt{2b^2 - a^2}.$$

Following up this idea Baur has solved the non-homogeneous system of an n-ary quadric and $n-1$ linear equations between the variables; viz. let the system be

$$a_{11}x_1^2 + \dots + 2a_{ik}x_i x_k + \dots = u \quad \dots\dots\dots\dots(1),$$

$$\left.\begin{array}{l} c_{11}x_1 + \dots + c_{1n}x_n \quad = y_1 \\ c_{21}x_1 + \dots + c_{2n}x_n \quad = y_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ c_{n-11}x_1 + \dots + c_{n-1n}x_n = y_{n-1} \end{array}\right\} \dots\dots\dots\dots (2).$$

Then we wish to establish a new linear equation

$$c_{n1}x_1 + \dots + c_{nn}x_n = y_n \dots\dots\dots\dots\dots\dots(3),$$

so that if we determine the values of $x_1 \dots x_n$ in terms of $y_1 \dots y_n$ from (2) and (3), and substitute their values in (1), the result shall only contain y_n in the form y_n^2. We are to have then

$$u = y_n^2 + \Sigma b_{ik} y_i y_k \qquad (i,\, k = 1,\, 2 \dots n-1) \dots\dots\dots(4).$$

Now if $$C = | c_{ik} |$$

we have $$Cx_i = C_{1i}y_1 + C_{2i}y_2 + \dots + C_{ni}y_n \quad \dots\dots\dots\dots(5).$$

Hence, differentiating (4) partially with respect to y_n, we get

$$2y_n = \frac{du}{dx_1}\frac{dx_1}{dy_n} + \frac{du}{dx_2}\frac{dx_2}{dy_n} + \dots + \frac{du}{dx_n}\frac{dx_n}{dy_n},$$

or, by aid of (5), if $$u_i = \tfrac{1}{2}\frac{du}{dx_i},$$

$$Cy_n = u_1 C_{n1} + u_2 C_{n2} + \ldots + u_n C_{nn}$$

$$= \begin{vmatrix} c_{11}, & c_{12} & \ldots & c_{1n} \\ c_{21}, & c_{22} & \ldots & c_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ c_{n-11}, & c_{n-12} & \ldots & c_{n-1n} \\ u_1, & u_2 & \ldots & u_n \end{vmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

Substituting for the differential coefficients their values we determine the form of the equation (3). We have still to determine the value of y_n. To do this we introduce the $n(n-1)$ quantities

$$\begin{matrix} e_{11}, & e_{12} & \ldots & e_{1n} \\ e_{21}, & e_{22} & \ldots & e_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ e_{n-11}, & e_{n-12} & \ldots & e_{n-1n}, \end{matrix}$$

such that $$e_{r1}a_{1k} + e_{r2}a_{2k} + \ldots + e_{rn}a_{nk} = c_{rk};$$

and hence $$Ae_{ri} = c_{r1}A_{i1} + c_{r2}A_{i2} + \ldots + c_{rn}A_{in},$$

where $$A = |a_{ik}|.$$

Thus

$$A\begin{vmatrix} e_{11}, & e_{12} & \ldots & e_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ e_{n-11}, & e_{n-12} & \ldots & e_{n-1n} \\ x_1, & x_2 & \ldots & x_n \end{vmatrix} = \begin{vmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ c_{n-11} & \ldots & c_{n-1n} \\ u_1 & \ldots & u_n \end{vmatrix} = Cy_n \ldots\ldots(7).$$

Now from the product of (6) and (7)

$$C^2 y_n^2 = A\begin{vmatrix} e_{11} & \ldots & e_{1n} \\ \ldots & \ldots & \ldots \\ e_{n-11} & \ldots & e_{n-1n} \\ x_1 & \ldots & x_n \end{vmatrix} \cdot \begin{vmatrix} c_{11} & \ldots & c_{1n} \\ \ldots & \ldots & \ldots \\ c_{n-11} & \ldots & c_{n-1n} \\ u_1 & \ldots & u_n \end{vmatrix}$$

$$= A\begin{vmatrix} B_{11}, & B_{12} & \ldots & B_{1n-1}, & y_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ B_{n-11}, & B_{n-12} & \ldots & B_{n-1n-1}, & y_{n-1} \\ y_1, & y_2 & \ldots & y_{n-1}, & u \end{vmatrix} \qquad \ldots\ldots\ldots\ldots(8),$$

where
$$B_{rs} = c_{r1}e_{s1} + c_{r2}e_{s2} + \dots + c_{rn}e_{sn},$$
$$\begin{aligned} AB_{rs} &= c_{r1}(c_{s1}A_{11} + c_{s2}A_{12} + \dots) \\ &\quad + c_{r2}(c_{s1}A_{21} + c_{s2}A_{22} + \dots) \\ &\quad + \dots \\ &= -\begin{vmatrix} 0, & c_{r1}, & c_{r2} & \dots & c_{rn} \\ c_{s1}, & a_{11}, & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ c_{sn}, & a_{n1}, & a_{n2} & \dots & a_{nn} \end{vmatrix} = AB_{sr}. \end{aligned}$$

On the right-hand side of (8) all the quantities are known from (1) and (2). Thus Cy_n is known; substitute its value in the left of (6) and we have the required equation (3), which with the equations (2) forms a system of n linear equations sufficient to determine the quantities $x_1 \dots x_n$.

19. The equation
$$\begin{vmatrix} a_{11} - \lambda, & a_{12}, & a_{13} & \dots & a_{1n} \\ a_{21}, & a_{22} - \lambda, & a_{23} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2}, & a_{n3} & \dots & a_{nn} - \lambda \end{vmatrix} = 0$$
(where $a_{ik} = a_{ki}$) formed by taking λ from each of the leading elements of a symmetrical determinant is of considerable importance in analysis. The following proof that its roots are real, when the quantities a_{ik} are real, is due to Sylvester. If we denote the left-hand side of the equation by $\phi(\lambda)$ we have
$$\phi(-\lambda) = \begin{vmatrix} a_{11} + \lambda, & a_{12} & \dots & a_{1n} \\ a_{21}, & a_{22} + \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1}, & a_{n2} & \dots & a_{nn} + \lambda \end{vmatrix}$$
and hence
$$\phi(\lambda)\,\phi(-\lambda) = \begin{vmatrix} c_{11} - \lambda^2, & c_{12} & \dots & c_{1n} \\ c_{21}, & c_{22} - \lambda^2 & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}, & c_{n2} & \dots & c_{nn} - \lambda^2 \end{vmatrix}$$
where
$$c_{rs} = a_{r1}a_{s1} + a_{r2}a_{s2} + \dots + a_{rn}a_{sn};$$

the λ disappears, because $a_{rs} = a_{sr}$. Hence, expanding the right-hand side by Art. 23 of Chap. IV.,

$$\phi(\lambda)\,\phi(-\lambda) = C - \lambda^2 \Sigma C_1 + \lambda^4 \Sigma C_2 - \ldots + (-\lambda^2)^n.$$

Now, by v. 9, C_1, $C_2 \ldots$ are all sums of squares, so that the coefficient of each power of λ is positive. Hence, if we equate the right-hand side of this last equation to zero, Des Cartes' rule shews that it cannot have a negative root. Thus λ cannot be of the form $\beta\sqrt{-1}$. In order to shew that it cannot have the form $\alpha + \beta\sqrt{-1}$ we have only to write $a_{11} - \alpha = a_{11}'$, &c., and the case is reduced to the preceding.

CHAPTER XII.

RATIONAL FUNCTIONAL DETERMINANTS.

1. If we have a series of n quantities $x, y, z \ldots u, t$ we shall denote the product of all the $\frac{1}{2}n(n-1)$ differences, obtained by subtracting from each number all that follow it, by

$$\zeta^{\frac{1}{2}}(x, y, z \ldots u, t).$$

So that

$$\begin{aligned}\zeta^{\frac{1}{2}}(x, y, z \ldots u, t) = (x-y)(x-z) & \ldots (x-t) \\ (y-z) & \ldots (y-t) \\ & \cdots\cdots\cdots\cdots \\ & (u-t).\end{aligned}$$

This function $\zeta^{\frac{1}{2}}(x, y, z \ldots u, t)$ is an alternating function of all the quantities $x, y, z \ldots t$; viz. on interchanging any two of these it changes its sign, but not its absolute magnitude. It is thus of the nature of a square root, having two values equal in absolute magnitude, but opposite in sign. This is conveniently indicated by the index $\frac{1}{2}$. The product of the squares of the differences will be denoted by $\zeta(x, y, z \ldots u, t)$, and is a symmetrical function. This notation is Sylvester's.

2. We have

$$\begin{vmatrix} x^{n-1}, & x^{n-2} \ldots x, & 1 \\ y^{n-1}, & y^{n-2} \ldots y, & 1 \\ \cdots & \cdots\cdots\cdots & \cdots \\ t^{n-1}, & t^{n-2} \ldots t, & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(x, y, z \ldots t).$$

For the determinant on the left vanishes if any two of the quantities $x, y \ldots t$ become equal, because then two rows become identical.

Thus the determinant divides by the difference between each pair of the letters, being a rational function. Hence it contains $\zeta^{\frac{1}{2}}(x, y \ldots t)$ as a factor. But the leading term in the determinant is $x^{n-1}y^{n-2} \ldots u \,.\, 1$, which is also a term in $\zeta^{\frac{1}{2}}(x \ldots t)$ with its proper sign. Thus the theorem follows.

3. By a similar argument it may be proved that if a rational integral function of $x \ldots t$ changes sign when any two of the variables are interchanged, it is divisible by $\zeta^{\frac{1}{2}}(x \ldots t)$.

4. If $f_i(x)$ be a function of the ith degree in x, the coefficient of whose highest term is unity, we have

$$\begin{vmatrix} f_{n-1}(x), & f_{n-2}(x) \ldots f_1(x), & 1 \\ f_{n-1}(y), & f_{n-2}(y) \ldots f_1(y), & 1 \\ \cdots\cdots & \cdots\cdots & \cdots \\ f_{n-1}(t), & f_{n-2}(t) \ldots f_1(t), & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(x, y \ldots t).$$

For if we subtract the last column, multiplied by a proper number, from the last but one, the elements in this column become $x, y \ldots t$. Now multiply the last two columns by the proper numbers, and subtract their sum from the last column but two, the elements of that column now become $x^2, y^2 \ldots t^2$. By proceeding in this way we reduce the determinant to that in Art. 2.

If the coefficients of the highest powers of x are not unity, the determinant is equal to $\zeta^{\frac{1}{2}}(x, y \ldots t)$ multiplied by the product of the highest coefficients in the separate functions.

For example, if

$$f_i(x) = \frac{x(x-1)\ldots(x-i+1)}{i\,!} = x_i,$$

$$\begin{vmatrix} x_{n-1}, & x_{n-2} \ldots x_1, & 1 \\ y_{n-1}, & y_{n-2} \ldots y_1, & 1 \\ \cdots & \cdots & \cdots \\ t_{n-1}, & t_{n-2} \ldots t_1, & 1 \end{vmatrix} = \frac{\zeta^{\frac{1}{2}}(x, y \ldots t)}{(n-1)\,!\,(n-2)\,!\ldots 2\,!}.$$

The denominator can also be written

$$2^{n-2} \,.\, 3^{n-3} \ldots (n-2)^2 \,.\, (n-1).$$

5. If $$f_i(x) = a_{1i}x^{n-1} + a_{2i}x^{n-2} + \dots + a_{ni},$$
we see by the theorem for multiplying two determinants (v. 4) that

$$\begin{vmatrix} f_1(x_1), & f_2(x_1) \dots f_n(x_1) \\ f_1(x_2), & f_2(x_2) \dots f_n(x_2) \\ \dots\dots & \dots\dots \\ f_1(x_n), & f_2(x_n) \dots f_n(x_n) \end{vmatrix} = \begin{vmatrix} a_{11} \dots a_{1n} \\ \dots\dots \\ a_{n1} \dots a_{nn} \end{vmatrix} \begin{vmatrix} x_1^{n-1}, & x_1^{n-2} \dots 1 \\ \dots\dots & \dots\dots \\ x_n^{n-1}, & x_n^{n-2} \dots 1 \end{vmatrix}$$
$$= | a_{nn} | \, \zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n).$$

If $f_i(x_k) = (x_k - y_i)^{n-1}$

$$| a_{nn} | = \begin{vmatrix} 1, & c_1(-y_1), & c_2(-y_1)^2 \dots (-y_1)^{n-1} \\ 1, & c_1(-y_2), & c_2(-y_2)^2 \dots (-y_2)^{n-1} \\ \dots & \dots\dots & \dots\dots \\ 1, & c_1(-y_n), & c_2(-y_n)^2 \dots (-y_n)^{n-1} \end{vmatrix}$$
$$= C\zeta^{\frac{1}{2}}(y_1, y_2 \dots y_n),$$

where C is the product of all the binomial coefficients of order $n-1$.

For the elements in each column of the determinant are multiplied by that power of -1, which is introduced by moving the column from its place in $\zeta^{\frac{1}{2}}$ to the place it occupies.

Thus

$$\begin{vmatrix} (x_1 - y_1)^{n-1}, & (x_1 - y_2)^{n-1} \dots (x_1 - y_n)^{n-1} \\ (x_2 - y_1)^{n-1}, & (x_2 - y_2)^{n-1} \dots (x_2 - y_n)^{n-1} \\ \dots\dots & \dots\dots \\ (x_n - y_1)^{n-1}, & (x_n - y_2)^{n-1} \dots (x_n - y_n)^{n-1} \end{vmatrix}$$
$$= C\,\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n)\,\zeta^{\frac{1}{2}}(y_1, y_2 \dots y_n).$$

If $x_i = y_i$ this gives us $\zeta(x_1 \dots x_n)$ in the form of a determinant.

6. We may give other determinant forms to the product
$$\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n)\,\zeta^{\frac{1}{2}}(y_1, y_2 \dots y_n).$$
Thus
$$\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n)\,\zeta^{\frac{1}{2}}(y_1, y_2 \dots y_n) = \begin{vmatrix} x_1^{n-1} \dots 1 \\ \dots\dots \\ x_n^{n-1} \dots 1 \end{vmatrix} \begin{vmatrix} y_1^{n-1} \dots 1 \\ \dots\dots \\ y_n^{n-1} \dots 1 \end{vmatrix}$$
$$= | c_{nn} |,$$

where, if we multiply by rows,

$$c_{ik} = x_i^{n-1} y_k^{n-1} + x_i^{n-2} y_k^{n-2} + \dots + x_i y_k + 1$$
$$= \frac{(x_i y_k)^n - 1}{x_i y_k - 1}.$$

Or if we multiply by columns

$$c_{ik} = x_1^{n-i} y_1^{n-k} + x_2^{n-i} y_2^{n-k} + \dots + x_n^{n-i} y_n^{n-k}.$$

If we put $x_i = y_i$ and $s_i = x_1^i + x_2^i + \dots + x_n^i$ we get

$$\zeta(x_1, x_2 \dots x_n) = \begin{vmatrix} s_{2n-2}, & s_{2n-3} & \dots & s_{n-1} \\ s_{2n-3}, & s_{2n-4} & \dots & s_{n-2} \\ \dots & \dots & \dots & \dots \\ s_{n-1}, & s_{n-2} & \dots & s_0 \end{vmatrix}$$
$$= \begin{vmatrix} s_0, & s_1 & \dots & s_{n-1} \\ s_1, & s_2 & \dots & s_n \\ \dots & \dots & \dots & \dots \\ s_{n-1}, & s_n & \dots & s_{2n-2} \end{vmatrix},$$

an orthosymmetrical determinant.

7. A more general theorem is the following. Consider the array

$$\begin{matrix} x_1^{m-1}, & x_1^{m-2} & \dots & x_1, & 1 \\ x_2^{m-1}, & x_2^{m-2} & \dots & x_2, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_n^{m-1}, & x_n^{m-2} & \dots & x_n, & 1, \end{matrix}$$

where n is greater than m. By compounding it with its conjugate, we get a determinant of the mth order which is equal to the sum of the squares of the n_m determinants, obtained by taking any m different rows in the array. The determinant has for elements

$$c_{ik} = x_1^{i-1} x_1^{k-1} + \dots + x_n^{i-1} x_n^{k-1}$$
$$= s_{i+k-2}.$$

Hence, by aid of Art. 6, we get

$$\Sigma \{\zeta(x_p, x_q \dots)\} = \begin{vmatrix} s_0, & s_1 & \dots & s_{m-1} \\ s_1, & s_2 & \dots & s_m \\ \dots & \dots & \dots & \dots \\ s_{m-1}, & s_m & \dots & s_{2m-2} \end{vmatrix},$$

where $x_p, x_q \dots$ are any m of the n quantities $x_1, x_2 \dots x_n$.

8. We have clearly by Art. 2

$$\begin{vmatrix} x^n, & x^{n-1} & \dots & x, & 1 \\ \alpha_1^n, & \alpha_1^{n-1} & \dots & \alpha_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n^n, & \alpha_n^{n-1} & \dots & \alpha_n, & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n) f(x)$$

where $f(x) = (x - \alpha_1)(x - \alpha_2) \dots (x - \alpha_n)$

$$= x^n - p_1 x^{n-1} + p_2 x^{n-2} - \dots + (-1)^{n-i} p_{n-i} x^i + \dots$$

By equating coefficients of x^i on both sides we get

$$\begin{vmatrix} \alpha_1^n & \dots & \alpha_1^{i+1}, & \alpha_1^{i-1} & \dots & 1 \\ \alpha_2^n & \dots & \alpha_2^{i+1}, & \alpha_2^{i-1} & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_n^n & \dots & \alpha_n^{i+1}, & \alpha_n^{i-1} & \dots & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n) p_{n-i},$$

where p_{n-i} is the sum of the products $n - i$ at a time, without repetition, of the quantities $\alpha_1 \dots \alpha_n$.

9. We may write the first identity of the preceding article in the form

$$\begin{vmatrix} \alpha_1^n, & \alpha_2^n & \dots & \alpha_n^n, & x^n, & 0 \\ \alpha_1^{n-1}, & \alpha_2^{n-1} & \dots & \alpha_n^{n-1}, & x^{n-1}, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_1, & \alpha_2 & \dots & \alpha_n, & x, & 0 \\ 1, & 1 & \dots & 1, & 1, & 0 \\ 0, & 0 & \dots & 0, & 0, & 1 \end{vmatrix} = (-1)^n \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n) f(x),$$

and similarly

$$\begin{vmatrix} \alpha_1^n, & \alpha_2^n & \dots & \alpha_n^n, & 0, & y^n \\ \alpha_1^{n-1}, & \alpha_2^{n-1} & \dots & \alpha_n^{n-1}, & 0, & y^{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_1, & \alpha_2 & \dots & \alpha_n, & 0, & y \\ 1, & 1 & \dots & 1, & 0, & 1 \\ 0, & 0 & \dots & 0, & 1, & 0 \end{vmatrix} = (-1)^{n+1} \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n) f(y).$$

Form the product of these two determinants by rows; thus

$$\begin{vmatrix} s_{2n}, & s_{2n-1} & \dots & s_n, & x^n \\ s_{2n-1}, & s_{2n-2} & \dots & s_{n-1}, & x^{n-1} \\ \dots & \dots & \dots & \dots & \dots \\ s_n, & s_{n-1} & \dots & s_0, & 1 \\ y^n, & y^{n-1} & \dots & 1, & 0 \end{vmatrix} = -\zeta(\alpha_1, \alpha_2 \dots \alpha_n) f(x) \cdot f(y),$$

from which by equating coefficients of the powers of x and y we get a number of theorems. s_r is now the sum of the rth powers of the roots of the equation $f(x)=0$.

10. We may extend the theorem of Art. 8 as follows: the value of the determinant

$$\begin{vmatrix} x_1^{n+r-1}, & x_1^{n+r-2} & \dots & x_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_r^{n+r-1}, & x_r^{n+r-2} & \dots & x_r, & 1 \\ \alpha_1^{n+r-1}, & \alpha_1^{n+r-2} & \dots & \alpha_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n^{n+r-1}, & \alpha_n^{n+r-2} & \dots & \alpha_n, & 1 \end{vmatrix}$$

which is of the form of that in Art. 2, may be expressed as the product of three factors.

First the product of the differences of all pairs of the quantities $x_1 \dots x_r$, i.e. $\zeta^{\frac{1}{2}}(x_1 \dots x_r)$, which by Art. 2 can be expressed as a determinant. Secondly, the product of the differences of all pairs of the quantities $\alpha_1 \dots \alpha_n$, i.e. $\zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n)$. And, lastly, the product of all such quantities as

$$\begin{aligned} f(x_i) &= (x_i-\alpha_1)(x_i-\alpha_2)\dots(x_i-\alpha_n) \\ &= x_i^n - p_1x_i^{n-1} + \dots + (-1)^{n-k}p_{n-k}x_i^k + \dots \end{aligned}$$

Hence its value is

$$\begin{vmatrix} x_1^{r-1}, & x_1^{r-2} & \dots & x_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ x_r^{r-1}, & x_r^{r-2} & \dots & x_r, & 1 \end{vmatrix} \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n) f(x_1) \dots f(x_r).$$

Multiplying the ith row by $f(x_i)$, and then equating coefficients of $x_1^u \,.\, x_2^v \,.\, x_3^w \dots$, we get the theorem:

If $D_{u,v,w\dots}$ is the determinant of order n formed by suppressing the columns containing the uth, vth, wth ... powers in the array

$$\begin{matrix} \alpha_1^{n+r-1}, & \alpha_1^{n+r-2} & \dots & \alpha_1, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \alpha_n^{n+r-1}, & \alpha_n^{n+r-2} & \dots & \alpha_n, & 1, \end{matrix}$$

then

$$D_{u,v,w\dots} = \begin{vmatrix} p_{n-u+r-1}, & p_{n-u+r-2} & \dots & p_{n-u} \\ p_{n-v+r-1}, & p_{n-v+r-2} & \dots & p_{n-v} \\ \dots & \dots & \dots & \dots \end{vmatrix} \zeta^{\frac{1}{2}}(\alpha_1, \ \alpha_2 \dots \alpha_n),$$

where p_k is the sum of the products k at a time of $\alpha_1 \dots \alpha_n$.

If k is negative or greater than n, $p_k = 0$, $p_0 = 1$.

11. Let us consider the determinant

$$D = \begin{vmatrix} \dfrac{1}{x_1 - \alpha_1}, & \dfrac{1}{x_1 - \alpha_2} & \cdots & \dfrac{1}{x_1 - \alpha_n} \\ \dfrac{1}{x_2 - \alpha_1}, & \dfrac{1}{x_2 - \alpha_2} & \cdots & \dfrac{1}{x_2 - \alpha_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{1}{x_n - \alpha_1}, & \dfrac{1}{x_n - \alpha_2} & \cdots & \dfrac{1}{x_n - \alpha_n} \end{vmatrix}.$$

Multiplying the ith row by

$$f(x_i) = u_i = (x_i - \alpha_1)(x_i - \alpha_2) \dots (x_i - \alpha_n),$$

we get

$$u_1 u_2 \dots u_n D = \left| \frac{u_i}{x_i - \alpha_k} \right|.$$

The determinant on the right is an integral and alternating function both of the quantities $x_1 \dots x_n$ and of $\alpha_1 \dots \alpha_n$. Hence by Art. 3 it divides by

$$\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n).$$

Comparing the orders of the determinant and this product we see they are the same, hence the additional factor is numerical only. To determine it, put $x_1, x_2 \dots x_n$ equal to $\alpha_1, \alpha_2 \dots \alpha_n$ respectively; then all the elements except those in the leading diagonal vanish, and

$$\begin{aligned} \frac{u_i}{x_i - \alpha_i} &= (x_i - \alpha_1)(x_i - \alpha_2) \dots (x_i - \alpha_{i-1})(x_i - \alpha_{i+1}) \dots (x_i - \alpha_n) \\ &= (-1)^{i-1}(\alpha_1 - \alpha_i) \dots (\alpha_{i-1} - \alpha_i)(\alpha_i - \alpha_{i+1}) \dots (\alpha_i - \alpha_n) \end{aligned}$$

when $x_i = \alpha_i$,

thus the determinant reduces to

$$(-1)^{\frac{n(n-1)}{2}} \zeta(\alpha_1, \alpha_2 \dots \alpha_n),$$

which determines the factor. Hence

$$D = \frac{(-1)^{\frac{n(n-1)}{2}} \zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n)}{u_1 u_2 \dots u_n}.$$

12. If D_{ik} is the complement of $\dfrac{1}{x_i - \alpha_k}$ in the determinant D, then D_{ik} is equal to the expression obtained by omitting x_i and α_k on the right, and multiplying by $(-1)^{i+k}$.

$$\therefore D_{ik} = (-1)^{i+k} \frac{(-1)^{\frac{n(n-1)}{2}} \zeta^{\frac{1}{2}}(x_1 \ldots x_{i-1} x_{i+1} \ldots x_n) \zeta^{\frac{1}{2}}(\alpha_1 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_n)}{v_1 v_2 \ldots v_{n-1}},$$

where

$$v_1 v_2 \ldots v_{n-1} = \frac{u_1}{x_1 - \alpha_k} \cdot \frac{u_2}{x_2 - \alpha_k} \cdots \frac{u_{i-1}}{x_{i-1} - \alpha_k} \cdot \frac{u_{i+1}}{x_{i+1} - \alpha_k} \cdots \frac{u_n}{x_n - \alpha_k}.$$

Now if we write

$$g(z) = (z - x_1)(z - x_2) \ldots (z - x_n)$$

$$\zeta^{\frac{1}{2}}(x_1 \ldots x_{i-1} x_{i+1} \ldots x_n) \zeta^{\frac{1}{2}}(\alpha_1 \ldots \alpha_{k-1} \alpha_{k+1} \ldots \alpha_n) = \frac{(-1)^{i+k} \zeta^{\frac{1}{2}}(x_1 \ldots x_n) \zeta^{\frac{1}{2}}(\alpha_1 \ldots \alpha_n)}{g'(x_i) f'(\alpha_k)}$$

$$(x_1 - \alpha_k)(x_2 - \alpha_k) \ldots (x_{i-1} - \alpha_k)(x_{i+1} - \alpha_k) \ldots (x_n - \alpha_k) = (-1)^n \frac{g(\alpha_k)}{x_i - \alpha_k},$$

then
$$\frac{D_{ik}}{D} = -\frac{f(x_i)\, g(\alpha_k)}{f'(\alpha_k)\, g'(x_i)} \cdot \frac{1}{x_i - \alpha_k}.$$

13. The preceding article enables us to solve the system of equations

$$\frac{y_1}{x_1 - \alpha_1} + \frac{y_2}{x_1 - \alpha_2} + \ldots + \frac{y_n}{x_1 - \alpha_n} = u_1$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\frac{y_1}{x_n - \alpha_1} + \frac{y_2}{x_n - \alpha_2} + \ldots + \frac{y_n}{x_n - \alpha_n} = u_n,$$

viz.
$$y_k = -\frac{g(\alpha_k)}{f'(\alpha_k)} \left\{ \frac{f(x_1)}{g'(x_1)} \frac{u_1}{x_1 - \alpha_k} + \ldots + \frac{f(x_n)}{g'(x_n)} \frac{u_n}{x_n - \alpha_k} \right\}.$$

In particular, suppose that $u_1 = u_2 = \ldots = u_n$. Then since by the rule for resolving a rational fraction into partial fractions

$$\frac{f(x)}{g(x)} = \frac{(x - \alpha_1)(x - \alpha_2) \ldots (x - \alpha_n)}{(x - x_1)(x - x_2) \ldots (x - x_n)}$$

$$= 1 + \Sigma \frac{f(x_i)}{g'(x_i)} \frac{1}{x - x_i},$$

we see by putting $x = \alpha_k$ in this, that

$$\frac{f(x_1)}{g'(x_1)}\frac{1}{x_1-\alpha_k}+\ldots+\frac{f(x_n)}{g'(x_n)}\frac{1}{x_n-\alpha_k}=1.$$

Hence if $u_i = 1$, $\qquad y_k = -\dfrac{g(\alpha_k)}{f'(\alpha_k)}$.

14. If in the determinant D of Art. 11 we expand each term in a series as follows

$$\frac{1}{x_i-\alpha_k}=\frac{1}{x_i}+\frac{\alpha_k}{x_i^2}+\ldots+\frac{\alpha_k{}^p}{x_i{}^{p+1}}+\ldots$$

we see that the term in the expansion of the determinant which multiplies $(x_1{}^{p+1}\,.\,x_2{}^{q+1}\ldots x_n{}^{s+1})^{-1}$ is

$$\begin{vmatrix} \alpha_1{}^p, & \alpha_2{}^p & \ldots & \alpha_n{}^p \\ \alpha_1{}^q, & \alpha_2{}^q & \ldots & \alpha_n{}^q \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_1{}^s, & \alpha_2{}^s & \ldots & \alpha_n{}^s \end{vmatrix}.$$

To expand the right-hand side of the identity at the end of Art. 11, we have

$$\frac{1}{u_i}=\frac{1}{(x_i-\alpha_1)(x_i-\alpha_2)\ldots(x_i-\alpha_n)}$$
$$=\frac{1}{x_i^n}+\frac{H_1}{x_i^{n+1}}+\ldots+\frac{H_r}{x_i^{n+r}}+\ldots$$

Here H_r is the sum of all the homogeneous powers and products of order r, which can be formed from the quantities $\alpha_1, \alpha_2 \ldots \alpha_n$.

Now

$$\zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_n)=\begin{vmatrix} x_1^{n-1}, & x_1^{n-2} & \ldots & x_1, & 1 \\ x_2^{n-1}, & x_2^{n-2} & \ldots & x_2, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_n^{n-1}, & x_n^{n-2} & \ldots & x_n, & 1 \end{vmatrix}.$$

Multiply the ith row of this determinant by the expansion of u_i^{-1}; the coefficient of $(x_1{}^{p+1}\,.\,x_2{}^{q+1}\ldots x_n{}^{s+1})^{-1}$ is

$$\begin{vmatrix} H_p, & H_{p-1} & \ldots & H_{p+1-n} \\ H_q, & H_{q-1} & \ldots & H_{q+1-n} \\ \cdots & \cdots & \cdots & \cdots \\ H_s, & H_{s-1} & \ldots & H_{s+1-n} \end{vmatrix},$$

whence we get the final equation

$$\begin{vmatrix} \alpha_1^p, & \alpha_1^q \dots \alpha_1^s \\ \dots & \dots \\ \alpha_n^p, & \alpha_n^q \dots \alpha_n^s \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} \begin{vmatrix} H_p, & H_q & \dots & H_s \\ \dots & \dots & \dots & \dots \\ H_{p+1-n}, & H_{q+1-n} & \dots & H_{s+1-n} \end{vmatrix} \zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n),$$

with the convention that $H_0 = 1$, and $H_r = 0$ when r is negative.

15. As an example of Art. 14,

$$\begin{vmatrix} a^4, & a, & 1 \\ b^4, & b, & 1 \\ c^4, & c, & 1 \end{vmatrix} = - \begin{vmatrix} H_4, & H_1, & H_0 \\ H_3, & H_0, & 0 \\ H_2, & 0, & 0 \end{vmatrix} \begin{vmatrix} a^2, & a, & 1 \\ b^2, & b, & 1 \\ c^2, & c, & 1 \end{vmatrix}$$

$$= -(a^2 + b^2 + c^2 + bc + ca + ab)(b - c)(c - a)(a - b).$$

We may make use of the results of Arts. 14 and 10 to evaluate determinants whose elements are sines and cosines.

For example take

$$X = \begin{vmatrix} 1, & 1, & 1, & 1 \\ \cos A, & \cos B, & \cos C, & \cos D \\ \sin A, & \sin B, & \sin C, & \sin D \\ \sin 3A, & \sin 3B, & \sin 3C, & \sin 3D \end{vmatrix}.$$

Write for the sines and cosines their exponential values, and suppose $\epsilon^{iA} = a$, &c. Then, writing only the first column of the determinant,

$$X = -\frac{1}{2^3} \begin{vmatrix} 1 \\ a + a^{-1} \\ a - a^{-1} \\ a^3 - a^{-3} \end{vmatrix} = -\frac{1}{2^3 (abcd)^3} \begin{vmatrix} a^3 \\ a^4 + a^2 \\ a^4 - a^2 \\ a^6 - 1 \end{vmatrix}.$$

Add the second row to the third, divide by 2 and subtract the third row from the second, thus

$$X = -\frac{1}{4 (abcd)^3} \begin{vmatrix} a^3 \\ a^2 \\ a^4 \\ a^6 - 1 \end{vmatrix}.$$

Thus

$$4\,(abcd)^3 X = \begin{vmatrix} a^2 \\ a^3 \\ a^4 \\ a^6 \end{vmatrix} + \begin{vmatrix} 1 \\ a^2 \\ a^3 \\ a^4 \end{vmatrix},$$

where the first determinant

$$= a^2b^2c^2d^2 \begin{vmatrix} 1 \\ a \\ a^2 \\ a^4 \end{vmatrix} = a^2b^2c^2d^2\,(a-b)\,(a-c)\,(a-d)\,(a+b+c+d) \\ (b-c)\,(b-d) \\ (c-d)$$

by Art. 8. And the second, in like manner, is equal to

$$(a-b)\,(a-c)\,(a-d)\,(bcd+acd+abd+abc) \\ (b-c)\,(b-d) \\ (c-d).$$

Hence

$$X = \frac{(a-b)\,(a-c)\,(a-d)\,(b-c)\,(b-d)\,(c-d)}{4\,a^3b^3c^3d^3} \times$$

$$[a^2b^2c^2d^2\,(a+b+c+d) + abcd\,(a^{-1}+b^{-1}+c^{-1}+d^{-1})]$$

$$= \frac{1}{4} \cdot \frac{a-b}{\sqrt{ab}} \cdots \left[\sqrt{abcd}\,(a+b+c+d) + \frac{1}{\sqrt{abcd}}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)\right].$$

Hence if $$2S = A + B + C + D$$

$$X = -\,2^5\,.\,\Pi\,\sin\tfrac{1}{2}\,(A-B)\,[\cos(S+A)+\cos(S+B) \\ +\cos(S+C)+\cos(S+D)].$$

16. If we differentiate the determinant of Art. 11 with respect to x_i, the elements of the ith row become

$$\frac{-1}{(x_i-\alpha_1)^2},\ \frac{-1}{(x_i-\alpha_2)^2} \cdots \frac{-1}{(x_i-\alpha_n)^2}.$$

And thus

$$(-1)^n \frac{d^nD}{dx_1dx_2\ldots dx_n} = \left| \frac{1}{(x_i-\alpha_k)^2} \right| \\ = B.$$

We shall now shew that

$$\frac{B}{D} = \begin{Bmatrix} \dfrac{1}{x_1 - \alpha_1}, & \dfrac{1}{x_1 - \alpha_2} \cdots & \dfrac{1}{x_1 - \alpha_n} \\ \dfrac{1}{x_2 - \alpha_1}, & \dfrac{1}{x_2 - \alpha_2} \cdots & \dfrac{1}{x_2 - \alpha_n} \\ \cdots & \cdots & \cdots \\ \dfrac{1}{x_n - \alpha_1}, & \dfrac{1}{x_n - \alpha_2} \cdots & \dfrac{1}{x_n - \alpha_n} \end{Bmatrix},$$

where $\{\ \}$ means that the function on the right is to be formed like a determinant, only all the signs are positive instead of alternating.

Multiply the ith row of B by u_i^2, then

$$(u_1 u_2 \ldots u_n)^2 B = \left| \frac{u_i^2}{(x_i - \alpha_k)^2} \right| \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

The determinant on the right is an integral and alternating function, both of $x_1, x_2 \ldots x_n$ and of $\alpha_1, \alpha_2 \ldots \alpha_n$, hence it divides by

$$\zeta^{\frac{1}{2}}(x_1, x_2 \ldots x_n)\, \zeta^{\frac{1}{2}}(\alpha_1 \ldots \alpha_n).$$

If the quotient is $\phi(x_1, x_2 \ldots x_n)$, this is symmetrical with regard to each of the variables, and of order $n-1$. Thus

$$\frac{B}{D} = (-1)^{\frac{n(n-1)}{2}} \frac{\phi(x_1, x_2 \ldots x_n)}{u_1 u_2 \ldots u_n}.$$

Now, by repeated use of the rule for resolving a fraction into partial fractions,

$$\frac{\phi(x_1 \ldots x_n)}{f(x_1)} = \sum_i \frac{\phi(\alpha_i \ldots x_n)}{f'(\alpha_i)(x_1 - \alpha_i)},$$

$$\frac{\phi(\alpha_i, x_2 \ldots x_n)}{f(x_2)} = \sum_k \frac{\phi(\alpha_i, \alpha_k \ldots x_n)}{f'(\alpha_k)(x_2 - \alpha_k)},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

and we get finally

$$\frac{\phi(x_1, x_2 \ldots x_n)}{u_1 u_2 \ldots u_n}$$

$$= \Sigma \frac{\phi(\alpha_i, \alpha_k \ldots \alpha_p)}{f'(\alpha_i) f'(\alpha_k) \ldots f'(\alpha_p)(x_1 - \alpha_i)(x_2 - \alpha_k) \ldots (x_n - \alpha_p)} \ldots\ldots(2).$$

Now, in the first place, in the combination $i, k \dots p$, no repetition can occur, for in the product

$$\frac{B(u_1 \dots u_n)^2}{\zeta^{\frac{1}{2}}(x_1 \dots x_n)\,\zeta^{\frac{1}{2}}(\alpha_1 \dots \alpha_n)},$$

not only B, but also $\dfrac{\{f(x_1)\}^2}{x_2 - x_1}$ vanishes if x_1 and x_2 both coincide with α_h. Hence on the right of (2) we must write for $i, k \dots p$ all permutations of $1, 2 \dots n$.

Now if we write $\alpha_i, \alpha_k \dots \alpha_p$ for $x_1, x_2 \dots x_n$ respectively, only a single term of $(u_1 \dots u_n)^2 B$ remains, viz.

$$\pm\,[f'(\alpha_i) f'(\alpha_k) \dots f'(\alpha_p)]^2,$$

while

$$\begin{aligned}\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_n) &= \zeta^{\frac{1}{2}}(\alpha_i, \alpha_k \dots \alpha_p)\\ &= \pm\,\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n),\end{aligned}$$

the ambiguous sign being the same for both. Thus

$$\begin{aligned}\phi(\alpha_i, \alpha_k \dots \alpha_p) &= \frac{[f'(\alpha_i) f'(\alpha_k) \dots f'(\alpha_p)]^2}{\zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n)}\\ &= (-1)^{\frac{n(n-1)}{2}} f'(\alpha_i) f'(\alpha_k) \dots f'(\alpha_p).\end{aligned}$$

Thus

$$\frac{B}{D} = \Sigma \frac{1}{(x_1 - \alpha_i)(x_2 - \alpha_k) \dots (x_n - \alpha_p)},$$

where $i, k \dots p$ is to be a permutation of $1, 2 \dots n$. This proves the theorem as stated at the beginning.

17. The coefficients in the expansion of the rational fraction

$$\frac{1 + b_1 x + b_2 x^2 + \dots}{1 + a_1 x + a_2 x^2 + \dots},$$

in ascending powers of x can be represented as determinants. Viz. if the expansion is

$$1 + P_1 x + P_2 x^2 + \dots$$

we have

$$(1 + b_1 x + b_2 x^2 + \dots) = (1 + P_1 x + P_2 x^2 + \dots)(1 + a_1 x + a_2 x^2 + \dots),$$

and hence equating coefficients

$$\begin{aligned}
&P_1 &&= b_1 - a_1\\
&a_1 P_1 + P_2 &&= b_2 - a_2\\
&a_2 P_1 + a_1 P_2 + P_3 &&= b_3 - a_3\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&a_{n-1} P_1 + a_{n-2} P_2 + \ldots + P_n &&= b_n - a_n,
\end{aligned}$$

a system of equations to find P_n. The determinant of the system is unity. Hence if, after solving by III. 7, we move the last column to the first place and change the sign of this column,

$$P_n = (-1)^n \begin{vmatrix} a_1 - b_1, & 1 & & & \\ a_2 - b_2, & a_1, & 1 & & \\ a_3 - b_3, & a_2, & a_1, & 1 & \\ \cdots & \cdots & \cdots & \cdots & 1 \\ a_n - b_n, & a_{n-1}, & a_{n-2} & \cdots\cdots & a_1 \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} 1, & 1, & ., & ., & . & \\ b_1, & a_1, & 1, & ., & . & \\ b_2, & a_2, & a_1, & 1, & . & \\ b_3, & a_3, & a_2, & a_1, & 1 & \\ . & . & . & . & . & . \end{vmatrix},$$

as we see by subtracting the first column from the second in the latter determinant.

CHAPTER XIII.

ON JACOBIANS AND HESSIANS.

1. If $y_1, y_2 \dots y_n$ be n functions of the n independent variables $x_1, x_2 \dots x_n$, and if

$$a_{ik} = \frac{dy_i}{dx_k},$$

then the determinant $|a_{ik}|$ is called the Jacobian of the functions $y_1 \dots y_n$ with respect to the variables $x_1 \dots x_n$. The name was given by Sylvester after Jacobi, who first studied these functions.

The notations

$$\frac{d(y_1, y_2 \dots y_n)}{d(x_1, x_2 \dots x_n)}; \quad J(y_1, y_2 \dots y_n)$$

have been employed for Jacobians, each of which has its advantages. The first renders evident the remarkable analogy between Jacobians and ordinary differential coefficients. The second is useful when there is no doubt as to the independent variables.

If the y's are explicit functions, the Jacobian is formed by direct differentiation.

2. If the functions $y_1 \dots y_n$ are not independent, but are connected by an equation

$$\phi(y_1, y_2 \dots y_n) = 0,$$

the Jacobian vanishes. For if we differentiate this equation with respect to x_k, we get

$$\frac{d\phi}{dy_1}\frac{dy_1}{dx_k} + \frac{d\phi}{dy_2}\frac{dy_2}{dx_k} + \dots + \frac{d\phi}{dy_n}\frac{dy_n}{dx_k} = 0,$$

where $k = 1, 2 \ldots n$. Eliminating

$$\frac{d\phi}{dy_1}, \frac{d\phi}{dy_2} \cdots \frac{d\phi}{dy_n},$$

from these equations we get (XI. 2)

$$\frac{d(y_1, y_2 \ldots y_n)}{d(x_1, x_2 \ldots x_n)} = 0.$$

3. If the functions y are fractions with the same denominator, so that

$$y_i = \frac{u_i}{u},$$

$$u^2 \frac{dy_i}{dx_k} = u \frac{du_i}{dx_k} - u_i \frac{du}{dx_k}.$$

Thus

$$u^{2n+1} \frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} = \begin{vmatrix} u, & 0, & & 0 \\ u_1, & u\dfrac{du_1}{dx_1} - u_1\dfrac{du}{dx_1} & \ldots & u\dfrac{du_1}{dx_n} - u_1\dfrac{du}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ u_n, & u\dfrac{du_n}{dx_1} - u_n\dfrac{du}{dx_1} & \ldots & u\dfrac{du_n}{dx_n} - u_n\dfrac{du}{dx_n} \end{vmatrix}.$$

Add the first column multiplied by $\dfrac{du}{dx_i}$ to the $(i+1)$st column, and we get

$$u^{2n+1} \frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} = \begin{vmatrix} u, & u\dfrac{du}{dx_1} & \ldots & u\dfrac{du}{dx_n} \\ u_1, & u\dfrac{du_1}{dx_1} & \ldots & u\dfrac{du_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ u_n, & u\dfrac{du_n}{dx_1} & \ldots & u\dfrac{du_n}{dx_n} \end{vmatrix},$$

whence dividing each of the last n columns by u

$$\frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} = \frac{1}{u^{n+1}} \begin{vmatrix} u, & \dfrac{du}{dx_1} & \cdots & \dfrac{du}{dx_n} \\ u_1, & \dfrac{du_1}{dx_1} & \cdots & \dfrac{du_1}{dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ u_n, & \dfrac{du_n}{dx_1} & \cdots & \dfrac{du_n}{dx_n} \end{vmatrix}.$$

4. The determinant on the right has been denoted by $K(u, u_1 \dots u_n)$. It has interesting properties of its own. For example, since the Jacobian vanishes if the quantities $y_1 \dots y_n$ are related by an equation, it follows that

$$K(u, u_1 \dots u_n) = 0$$

if a homogeneous relation exists between $u, u_1 \dots u_n$.

If $$u_i = \frac{v_i}{t},$$

it is readily shewn that

$$K(u, u_1 \dots u_n) = \frac{1}{t^{n+1}} K(v, v_1 \dots v_n).$$

5. If the functions $y_1 \dots y_n$ possess a common factor, so that

$$y_i = u_i u,$$

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = \frac{1}{u} \begin{vmatrix} u, & 0, & & 0 \\ u_1, & u\dfrac{du_1}{dx_1} + u_1 \dfrac{du}{dx_1} & \dots & u\dfrac{du_1}{dx_n} + u_1\dfrac{du}{dx_n} \\ \dots & \dots & \dots & \dots \\ u_n, & u\dfrac{du_n}{dx_1} + u_n\dfrac{du}{dx_1} & \dots & u\dfrac{du_n}{dx_n} + u_n\dfrac{du}{dx_n} \end{vmatrix}.$$

In this determinant multiply the first column by $\dfrac{du}{dx_i}$, and subtract it from the $(i+1)$st column, then

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = u^{n-1} \begin{vmatrix} u, & -\dfrac{du}{dx_1} & \dots & -\dfrac{du}{dx_n} \\ u_1, & \dfrac{du_1}{dx_1} & \dots & \dfrac{du_1}{dx_n} \\ \dots & \dots & \dots & \dots \\ u_n, & \dfrac{du_n}{dx_1} & \dots & \dfrac{du_n}{dx_n} \end{vmatrix}$$

$$= 2u^n \frac{d(u_1, u_2 \dots u_n)}{d(x_1 \dots x_n)} - u^{n-1} K(u, u_1 \dots u_n).$$

6. If the functions $y_1 \dots y_n$ are given only as implicit functions of $x_1 \dots x_n$ by means of the n equations

$$F_1(y_1 \dots y_n, x_1 \dots x_n) = 0, \dots F_n(y_1 \dots y_n, x_1 \dots x_n) = 0$$

then

$$\frac{d\,(y_1 \dots y_n)}{d\,(x_1 \dots x_n)} = (-1)^n \frac{d\,(F_1 \dots F_n)}{d\,(x_1 \dots x_n)} \div \frac{d\,(F_1 \dots F_n)}{d\,(y_1 \dots y_n)}.$$

For if we differentiate the ith of the given equations with respect to x_k we get

$$\frac{dF_i}{dy_1}\frac{dy_1}{dx_k} + \frac{dF_i}{dy_2}\frac{dy_2}{dx_k} + \dots + \frac{dF_i}{dy_n}\frac{dy_n}{dx_k} = -\frac{dF_i}{dx_k}.$$

Thus by the rule for multiplying two determinants (v. 4)

$$(-1)^n \left|\frac{dF_i}{dx_k}\right| = \left|\frac{dF_i}{dy_k}\right| \cdot \left|\frac{dy_i}{dx_k}\right|,$$

or
$$(-1)^n \frac{d\,(F_1 \dots F_n)}{d\,(x_1 \dots x_n)} = \frac{d\,(F_1 \dots F_n)}{d\,(y_1 \dots y_n)} \cdot \frac{d\,(y_1 \dots y_n)}{d\,(x_1 \dots x_n)},$$

which proves the theorem.

(i) If F_i does not contain $x_1 \dots x_{i-1}$, then in the determinant

$$\frac{d\,(F_1 \dots F_n)}{d\,(x_1 \dots x_n)}$$

all elements below the leading diagonal vanish, and it reduces to

$$\frac{dF_1}{dx_1} \cdot \frac{dF_2}{dx_2} \cdots \frac{dF_n}{dx_n}.$$

(ii) If
$$F_i = -y_i + f_i\,(x_1 \dots x_n),$$

then
$$\frac{d\,(F_1 \dots F_n)}{d\,(y_1 \dots y_n)} = (-1)^n,$$

and
$$\frac{d\,(y_1 \dots y_n)}{d\,(x_1 \dots x_n)} = \frac{d\,(f_1 \dots f_n)}{d\,(x_1 \dots x_n)}.$$

(iii) Suppose that from the given system we deduce by elimination

$$\begin{aligned}
y_1 &= \phi_1\,(x_1,\ x_2 \dots x_n) \\
y_2 &= \phi_2\,(y_1,\ x_2 \dots x_n) \\
y_3 &= \phi_3\,(y_1,\ y_2,\ x_3 \dots x_n) \\
&\dots\dots\dots\dots\dots\dots \\
y_n &= \phi_n\,(y_1 \dots y_{n-1},\ x_n).
\end{aligned}$$

Since

$$\frac{d\phi_i}{dy_1}\frac{dy_1}{dx_k} + \dots + \frac{d\phi_i}{dy_{i-1}}\frac{dy_{i-1}}{dx_k} + \frac{d\phi_i}{dx_k} = \frac{dy_i}{dx_k},$$

we have

$$\begin{vmatrix} \frac{d\phi_1}{dx_1}, & \frac{d\phi_1}{dx_2}, & \frac{d\phi_1}{dx_3} \dots \\ 0, & \frac{d\phi_2}{dx_2}, & \frac{d\phi_2}{dx_3} \dots \\ 0, & 0, & \frac{d\phi_3}{dx_3} \dots \\ \dots & \dots & \dots \end{vmatrix} = \begin{vmatrix} 1, & 0, & 0 \dots \\ -\frac{d\phi_2}{dy_1}, & 1, & 0 \dots \\ -\frac{d\phi_3}{dy_1}, & -\frac{d\phi_3}{dy_2}, & 1 \dots \\ \dots & \dots & \dots \end{vmatrix} \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)}.$$

It follows then that

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = \frac{d\phi_1}{dx_1} \cdot \frac{d\phi_2}{dx_2} \cdots \frac{d\phi_n}{dx_n};$$

thus if $$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = 0,$$

we must have $$\frac{d\phi_1}{dx_1} \frac{d\phi_2}{dx_2} \cdots \frac{d\phi_n}{dx_n} = 0,$$

i.e. we must have $$\frac{d\phi_i}{dx_i} = 0,$$

where i is some number between 1 and n. Hence ϕ_i does not contain x_i. That is to say, we have

$$y_i = \phi_i(y_1 \dots y_{i-1},\ x_{i+1} \dots x_n).$$

Now $$y_{i+1} = \phi_{i+1}(y_1 \dots y_i,\ x_{i+1} \dots x_n),$$

and by eliminating x_{i+1} between these we obtain

$$y_{i+1} = \psi_{i+1}(y_1 \dots y_i,\ x_{i+2} \dots x_n),$$

so that y_{i+1} does not contain x_{i+1}. Similarly we can shew that y_{i+2} does not contain x_{i+2}, and so on; finally y_n is independent of x_n or

$$y_n = \psi_n(y_1 \dots y_{n-1}).$$

So that if the Jacobian of $y_1 \dots y_n$ vanishes these functions are not independent. This is the converse of the theorem of Art. 2.

7. If $z_1 \dots z_n$ are functions of $y_1 \dots y_n$, and these again functions of $x_1 \dots x_n$; then

$$\frac{d(z_1 \dots z_n)}{d(x_1 \dots x_n)} = \frac{d(z_1 \dots z_n)}{d(y_1 \dots y_n)} \cdot \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)}.$$

For since

$$\frac{dz_i}{dx_k} = \frac{dz_i}{dy_1}\frac{dy_1}{dx_k} + \frac{dz_i}{dy_2}\frac{dy_2}{dx_k} + \dots + \frac{dz_i}{dy_n}\frac{dy_n}{dx_k}$$

we have

$$\left|\frac{dz_i}{dx_k}\right| = \left|\frac{dz_i}{dy_k}\right| \times \left|\frac{dy_i}{dx_k}\right|,$$

which proves the theorem.

In like manner, if $z_1 \dots z_m$ are given as functions of $y_1 \dots y_n$, and these given as functions of $x_1 \dots x_m$; then

$$\frac{d(z_1 \dots z_m)}{d(x_1 \dots x_m)} = 0, \qquad \text{if } m > n.$$

But if $m < n$

$$\frac{d(z_1 \dots z_m)}{d(x_1 \dots x_m)} = \Sigma \frac{d(z_1, z_2 \dots z_m)}{d(y_t, y_u, y_v \dots)} \cdot \frac{d(y_t, y_u, y_v \dots)}{d(x_1, x_2 \dots x_m)},$$

where for $t, u, v \dots$ we take all m-ads in n (v. 3).

8. If $f_1 \dots f_n$ are independent functions of $x_1 \dots x_n$, then $x_1 \dots x_n$ are independent functions of $f_1 \dots f_n$, and we have

$$\frac{d(f_1 \dots f_n)}{d(x_1 \dots x_n)} \cdot \frac{d(x_1 \dots x_n)}{d(f_1 \dots f_n)} = 1.$$

For differentiating f_i with respect to f_k we must consider $x_1 \dots x_n$ to be functions of $f_1 \dots f_n$. Thus

$$\frac{df_i}{dx_1}\frac{dx_1}{df_k} + \frac{df_i}{dx_2}\frac{dx_2}{df_k} + \dots + \frac{df_i}{dx_n}\frac{dx_n}{df_k}$$

is equal to unity or zero, according as k is or is not equal to i. Hence

$$\left|\frac{df_i}{dx_k}\right| \cdot \left|\frac{dx_i}{df_k}\right| = 1.$$

For in the product only the elements in the leading diagonal do not vanish, and these are all equal to unity.

9. If $$A = \left|\frac{df_i}{dx_k}\right|, \quad B = \left|\frac{dx_i}{df_k}\right|,$$

and A_{ik}, B_{ik} are the complements of $\frac{df_i}{dx_k}$ and $\frac{dx_i}{df_k}$, in these two determinants, we have

$$A\frac{dx_i}{df_k} = A_{ki}, \qquad B\frac{df_i}{dx_k} = B_{ki}.$$

Also
$$A\frac{d(x_1\ldots x_m)}{d(f_1\ldots f_m)}=\frac{d(f_{m+1}\ldots f_n)}{d(x_{m+1}\ldots x_n)},$$
$$B\frac{d(f_1\ldots f_m)}{d(x_1\ldots x_m)}=\frac{d(x_{m+1}\ldots x_n)}{d(f_{m+1}\ldots f_n)}.$$

For we have just seen that

$$\frac{df_1}{dx_1}\frac{dx_1}{df_k}+\frac{df_1}{dx_2}\frac{dx_2}{df_k}+\ldots+\frac{df_1}{dx_n}\frac{dx_n}{df_k}=0$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{df_k}{dx_1}\frac{dx_1}{df_k}+\frac{df_k}{dx_2}\frac{dx_2}{df_k}+\ldots+\frac{df_k}{dx_n}\frac{dx_n}{df_k}=1$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\frac{df_n}{dx_1}\frac{dx_1}{df_k}+\frac{df_n}{dx_2}\frac{dx_2}{df_k}+\ldots+\frac{df_n}{dx_n}\frac{dx_n}{df_k}=0.$$

Multiply these equations by A_{1i}, $A_{2i}\ldots A_{ni}$ respectively and add, then (IV. 11)

$$A\frac{dx_i}{df_k}=A_{ki}.$$

Similarly we can shew that

$$B\frac{df_i}{dx_k}=B_{ki}.$$

Again we have (VI. 5)

$$\begin{vmatrix} A_{11} & \ldots & A_{1m} \\ \ldots & \ldots & \ldots \\ A_{m1} & \ldots & A_{mm} \end{vmatrix}=A^{m-1}\frac{d(f_{m+1}\ldots f_n)}{d(x_{m+1}\ldots x_n)}.$$

Substitute in the left for A_{ik} the value just found; thus

$$A^m\frac{d(x_1\ldots x_m)}{d(f_1\ldots f_m)}=A^{m-1}\frac{d(f_{m+1}\ldots f_n)}{d(x_{m+1}\ldots x_n)},$$

which on dividing by A^{m-1} gives the result required.

The last equation is proved in a similar way.

10. If we suppose the functions $f_1\ldots f_n$ to depend on t, we have (IV. 16)

$$\frac{dA}{dt}=\Sigma A_{ik}\frac{d^2f_i}{dt\,dx_k}\qquad(i,\,k=1,\,2\ldots n),$$

and
$$A_{ik} = A \frac{dx_k}{df_i};$$

$$\therefore \frac{dA}{dt} = A\Sigma \left(\frac{d^2 f_i}{dt dx_1} \frac{dx_1}{df_i} + \frac{d^2 f_i}{dt dx_2} \frac{dx_2}{df_i} + \dots\right)$$

$$= A\Sigma \frac{d}{df_i}\left(\frac{df_i}{dt}\right),$$

or
$$\frac{d}{dt} \cdot \log A = \Sigma \frac{d}{df_i}\left(\frac{df_i}{dt}\right).$$

A similar relation holds for B.

11. The relations between Jacobians present great resemblance to the ordinary formulæ in the differential calculus.

Thus the formulæ

$$\frac{d(z_1 \dots z_n)}{d(x_1 \dots x_n)} = \frac{d(z_1 \dots z_n)}{d(y_1 \dots y_n)} \cdot \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)},$$

$$\frac{d(f_1 \dots f_n)}{d(x_1 \dots x_n)} \cdot \frac{d(x_1 \dots x_n)}{d(f_1 \dots f_n)} = 1,$$

are the analogues of

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx} \text{ and } \frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

This analogy, which was perceived by Jacobi, led Bertrand to devise a new definition of a Jacobian. Let $f_1 \dots f_n$ be n functions of the variables $x_1 \dots x_n$. Now if we give to the variables n distinct series of increments

$$\begin{matrix} d_1 x_1, & d_1 x_2 & \dots & d_1 x_n \\ d_2 x_1, & d_2 x_2 & \dots & d_2 x_n \\ \dots & \dots & \dots & \dots \\ d_n x_1, & d_n x_2 & \dots & d_n x_n \end{matrix} \qquad (1),$$

let the corresponding increments of the functions be

$$\begin{matrix} d_1 f_1, & d_1 f_2 & \dots & d_1 f_n \\ d_2 f_1, & d_2 f_2 & \dots & d_2 f_n \\ \dots & \dots & \dots & \dots \\ d_n f_1, & d_n f_2 & \dots & d_n f_n \end{matrix} \qquad (2).$$

Then just as the differential coefficient of a single function of a single variable is defined to be the limiting ratio of corresponding incre-

ments of the function and variable; the Jacobian of the functions $f_1 \dots f_n$ of the n variables $x_1 \dots x_n$ is defined to be the limiting ratio of the determinants of the systems of increments (2) and (1). That this leads to the same Jacobian as before is plain from the equation

$$d_k f_i = \frac{df_i}{dx_1} d_k x_1 + \frac{df_i}{dx_2} d_k x_2 + \dots + \frac{df_i}{dx_n} d_k x_n,$$

which gives (v. 4)

$$|d_k f_i| = |d_k x_i| \cdot \left| \frac{df_i}{dx_k} \right|,$$

or

$$\frac{|d_k f_i|}{|d_k x_i|} = \frac{d(f_1 \dots f_n)}{d(x_1 \dots x_n)},$$

according to our former definition.

Using this new definition we can prove all our former theorems. Let us use it to prove the first of the above equations, viz. the theorem of Art. 7. If the system of increments given to $x_1 \dots x_n$ be

$$\begin{array}{c} d_1 x_1 \dots d_1 x_n \\ \dots\dots\dots\dots \\ d_n x_1 \dots d_n x_n, \end{array}$$

let the corresponding systems for $y_1 \dots y_n$ and $z_1 \dots z_n$ be

$$\begin{array}{cc} d_1 y_1 \dots d_1 y_n & d_1 z_1 \dots d_1 z_n \\ \dots\dots\dots\dots & \dots\dots\dots\dots \\ d_n y_1 \dots d_n y_n & d_n z_1 \dots d_n z_n. \end{array}$$

Then we have identically

$$\frac{|d_i z_k|}{|d_i x_k|} = \frac{|d_i z_k|}{|d_i y_k|} \cdot \frac{|d_i y_k|}{|d_i x_k|};$$

or by definition,

$$\frac{d(z_1 \dots z_n)}{d(x_1 \dots x_n)} = \frac{d(z_1 \dots z_n)}{d(y_1 \dots y_n)} \cdot \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)}.$$

12. We can also, using alternate numbers, obtain a symbolic expression for the Jacobian, from which the ordinary results follow. Viz., $y_1 \dots y_n$, being n functions of $x_1 \dots x_n$, let

$$y = e_1 y_1 + e_2 y_2 + \dots + e_n y_n,$$

$$x = e_1 x_1 + e_2 x_2 + \dots + e_n x_n.$$

Then

$$\frac{dy}{dx_i} = e_1 \frac{dy_1}{dx_i} + e_2 \frac{dy_2}{dx_i} + \ldots + e_n \frac{dy_n}{dx_i},$$

whence (II. 15)

$$\frac{dy}{dx_1} \cdot \frac{dy}{dx_2} \cdots \frac{dy}{dx_n} = \begin{vmatrix} \frac{dy_1}{dx_1} & \cdots & \frac{dy_1}{dx_n} \\ \cdots & \cdots & \cdots \\ \frac{dy_n}{dx_1} & \cdots & \frac{dy_n}{dx_n} \end{vmatrix}$$

$$= \frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} \quad \ldots\ldots\ldots\ldots(1).$$

But now

$$\frac{dy}{dx_i} = \frac{dy}{dx} \cdot \frac{dx}{dx_i}$$

$$= e_i \frac{dy}{dx}.$$

Thus the above equation (1) becomes

$$\left(\frac{dy}{dx}\right)^n = \frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)},$$

from which symbolical equation we can deduce our former theorems.

For example the equation

$$\left(\frac{dy}{dx}\right)^n \left(\frac{dx}{dy}\right)^n = 1$$

gives at once

$$\frac{d(y_1 \ldots y_n)}{d(x_1 \ldots x_n)} \cdot \frac{d(x_1 \ldots x_n)}{d(y_1 \ldots y_n)} = 1.$$

13. Jacobians occur in changing the variables in a multiple definite integral. Let us transform the integral

$$I = \iint \ldots F(y_1 \ldots y_n)\, dy_1 \ldots dy_n$$

to an integral with respect to $x_1 \ldots x_n$, the functions $y_1 \ldots y_n$ being supposed given functions of $x_1 \ldots x_n$.

We proceed in the manner used by Lagrange to transform a triple integral. Beginning with y_n we have to find the sum of the quantities

$$F dy_n,$$

while $y_1, y_2 \dots y_{n-1}$ remain constant. This gives us

$$0 = \frac{dy_1}{dx_1} dx_1 + \frac{dy_1}{dx_2} dx_2 + \dots + \frac{dy_1}{dx_n} dx_n$$

$$0 = \frac{dy_2}{dx_1} dx_1 + \frac{dy_2}{dx_2} dx_2 + \dots + \frac{dy_2}{dx_n} dx_n$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$dy_n = \frac{dy_n}{dx_1} dx_1 + \frac{dy_n}{dx_2} dx_2 + \dots + \frac{dy_n}{dx_n} dx_n.$$

Solving this to find dx_n we get (XI. 1)

$$J_{n-1} dy_n = J_n dx_n,$$

where $$J_r = \frac{d(y_1, y_2 \dots y_r)}{d(x_1, x_2 \dots x_r)}.$$

Hence we must replace dy_n by $\frac{J_n}{J_{n-1}} dx_n$, and

$$I = \int \dots F dy_1 \dots dy_n = \int \dots F \frac{J_n}{J_{n-1}} dy_1 \dots dy_{n-1} dx_n,$$

the limits of x_n being determined from those of y_n.

In this integral begin by integrating with respect to y_{n-1}. We have to find the sum of the quantities $F \frac{J_n}{J_{n-1}} dy_{n-1}$, while $y_1 \dots y_{n-2}, x_n$ remain constant, so that

$$0 = \frac{dy_1}{dx_1} dx_1 + \dots + \frac{dy_1}{dx_{n-1}} dx_{n-1}$$

$$0 = \frac{dy_2}{dx_1} dx_1 + \dots + \frac{dy_2}{dx_{n-1}} dx_{n-1}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$dy_{n-1} = \frac{dy_{n-1}}{dx_1} dx_1 + \dots + \frac{dy_{n-1}}{dx_{n-1}} dx_{n-1},$$

which gives

$$J_{n-2} dy_{n-1} = J_{n-1} dx_{n-1}.$$

Thus dy_{n-1} is to be replaced by $\frac{J_{n-1}}{J_{n-2}} dx_{n-1}$, and $F \frac{J_n}{J_{n-1}} dy_{n-1}$

by $F \cdot \frac{J_n}{J_{n-1}} \cdot \frac{J_{n-1}}{J_{n-2}} dx_{n-1}$. Hence, the limits being properly determined,
$$I = \int \dots F \frac{J_n}{J_{n-2}} dy_1 \dots dy_{n-2} dx_{n-1} dx_n.$$

Similarly if we began by integrating this with respect to y_{n-2} we should get a system of equations which would give us

$$dy_{n-2} = \frac{J_{n-2}}{J_{n-3}} dx_{n-2},$$

and

$$I = \int \dots F \frac{J_n}{J_{n-3}}, dy_1 \dots dy_{n-3} dx_{n-2} dx_{n-1} dx_n.$$

Proceeding in this way we should finally obtain

$$I = \int \dots F \frac{J_n}{J_1} dy_1 dx_2 \dots dx_n.$$

Then we integrate with respect to y_1, subject to the equations

$$dx_2 = 0, \ dx_3 = 0, \dots dx_n = 0,$$

so that we must replace dy_1 by $\frac{dy_1}{dx_1} dx_1$, i.e. $J_1 dx_1$.

Thus
$$\begin{aligned} I &= \int \dots F J_n dx_1 dx_2 \dots dx_n \\ &= \int \dots F(x) \frac{d(y_1, y_2 \dots y_n)}{d(x_1, x_2 \dots x_n)} dx_1 dx_2 \dots dx_n, \end{aligned}$$

$F(x)$ being the result of substituting in F for $y_1 \dots y_n$ their values in terms of $x_1 \dots x_n$.

14. As an example let us consider the following determinant of definite integrals due to Tissot; we shall however follow Enneper's proof.

Let $a_1, a_2 \dots a_n$ be n constant quantities in ascending order of magnitude, and let

$$\phi_m(x_m) = (x_m - a_1)^{p_1} (x_m - a_2)^{p_2} \dots (x_m - a_m)^{p_m} \\ (a_{m+1} - x_m)^{p_{m+1}} \dots (a_n - x_m)^{p_n},$$

where $p_1, p_2 \dots p_n$ are either positive proper fractions or any real

negative numbers. The determinant to be considered is then

$$D = \begin{vmatrix} J_{11} \dots J_{1n} \\ \dots\dots\dots\dots \\ J_{n1} \dots J_{nn} \end{vmatrix},$$

where

$$J_{ik} = \int_{a_k}^{a_k+1} \frac{x_k^{i-1} \epsilon^{-x_k}}{\phi_k(x_k)} dx_k$$

$(a_{n+1} = \infty)$.

Thus

$$D = (-1)^{\frac{n(n-1)}{2}} \int_{a_1}^{a_2} dx_1 \int_{a_2}^{a_3} dx_2 \dots \int_{a_n}^{\infty} dx_n \frac{\zeta^{\frac{1}{2}}(x_1 \dots x_n) \exp.(-x_1 - \dots - x_n)}{\phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)}$$

(exp. $u = \epsilon^u$).

Now let us introduce in place of $x_1, x_2 \dots x_n$ the n new variables $y_1 \dots y_n$, given by the equations

$$\frac{y_1}{x_1 - a_1} + \frac{y_2}{x_1 - a_2} + \dots + \frac{y_n}{x_1 - a_n} = 1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{y_1}{x_n - a_1} + \frac{y_2}{x_n - a_2} + \dots + \frac{y_n}{x_n - a_n} = 1.$$

Then by XII. 13,

$$y_k = -\frac{g(a_k)}{f'(a_k)};$$

and hence

$$\frac{dy_k}{dx_i} = \frac{y_k}{x_i - a_k}.$$

Thus by XII. 11,

$$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = y_1 \dots y_n \left| \frac{1}{x_i - a_k} \right|$$

$$= (-1)^{\frac{n(n-1)}{2}} \frac{y_1 \dots y_n \zeta^{\frac{1}{2}}(x_1 \dots x_n) \zeta^{\frac{1}{2}}(a_1 \dots a_n)}{f(x_1) f(x_2) \dots f(x_n)}.$$

Now $$y_1 \dots y_n = \frac{g(a_1) \dots g(a_n)}{\zeta(a_1 \dots a_n)} (-1)^{\frac{n(n+1)}{2}};$$

therefore $$\frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)} = \frac{\zeta^{\frac{1}{2}}(x_1 \dots x_n)}{\zeta^{\frac{1}{2}}(a_1 \dots a_n)}.$$

Hence in the integral we replace $dx_1 \dots dx_n \zeta^{\frac{1}{2}}(x_1 \dots x_n)$ by $dy_1 \dots dy_n \zeta^{\frac{1}{2}}(a_1 \dots a_n)$.

Now if we write

$$F_m(z) = (z - a_1) \dots (z - a_m)(a_{m+1} - z) \dots (a_n - z)$$

we have

$$y_1^{p_1} y_2^{p_2} \dots y_n^{p_n} = \frac{\phi_1(x_1)\phi_2(x_2)\dots\phi_n(x_n)}{F_1'(a_1)^{p_1} F_2'(a_2)^{p_2} \dots F_n'(a_n)^{p_n}}.$$

Hence

$$\frac{\zeta^{\frac{1}{2}}(x_1 \dots x_n)}{\phi_1(x_1)\dots\phi_n(x_n)} dx_1 \dots dx_n$$

is replaced by

$$\frac{dy_1 \dots dy_n}{y_1^{p_1} \dots y_n^{p_n}} \frac{\zeta^{\frac{1}{2}}(a_1 \dots a_n)}{F_1'(a_1)^{p_1} \dots F_n'(a_n)^{p_n}}.$$

Again $x_1 \dots x_n$ can be regarded as the roots of the equation

$$\frac{y_1}{z - a_1} + \frac{y_2}{z - a_2} + \dots + \frac{y_n}{z - a_n} = 1,$$

the roots of which lie between a_1 and a_2; a_2 and a_3; ... a_n and ∞.

Hence $y_1 \dots y_n$ take all positive real values. Also we have

$$x_1 + x_2 + \dots + x_n = y_1 + y_2 + \dots + y_n + a_1 + \dots + a_n.$$

Thus our integral reduces to

$$\frac{(-1)^{\frac{n(n-1)}{2}} \zeta^{\frac{1}{2}}(a_1 \dots a_n) \exp.(-a_1 - \dots - a_n)}{F_1'(a_1)^{p_1} \dots F_n'(a_n)^{p_n}} \times$$

$$\int_0^\infty \dots \frac{\exp.(-y_1 - \dots - y_n)}{y_1^{p_1} \dots y_n^{p_n}} dy_1 \dots dy_n$$

$$= \frac{(-1)^{\frac{n(n-1)}{2}} \Gamma(1-p_1)\Gamma(1-p_2)\dots\Gamma(1-p_n)}{\{F_1'(a_1)^{2p_1-1} \dots F_n'(a_n)^{2p_n-1}\}^{\frac{1}{2}}} \epsilon^{-a_1-a_2-\dots-a_n}.$$

15. If u be a function of n variables $x_1, x_2 \dots x_n$ and $y_1 \dots y_n$ its differential coefficients with respect to these variables, since

$$\frac{dy_i}{dx_k} = \frac{d}{dx_k}\left(\frac{du}{dx_i}\right) = \frac{d^2u}{dx_k dx_i} = u_{ki} = u_{ik},$$

the Jacobian of $y_1 \dots y_n$ is a symmetrical determinant formed from the second differential coefficients of u. This determinant is called the Hessian of u (after Hesse), and is denoted by $H(u)$, so that

$$H(u) = |u_{ik}|.$$

The Hessian of u will vanish if the first differential coefficients of u are not independent (Art. 2).

For example, if

$$u = x_1^2 x_2^2 + x_1^2 x_3^2 + \dots + x_i^2 x_k^2 + \dots + x^2_{n-1} x_n^2,$$

$$\frac{d^2 u}{dx_i^2} = 2\left(x_1^2 + \dots + x^2_{i-1} + x^2_{i+1} + \dots + x_n^2\right),$$

$$\frac{d^2 u}{dx_i dx_k} = 4x_i x_k;$$

$$\therefore\ H(u) = \begin{vmatrix} 2\left(x_2^2 + x_3^2 + \dots + x_n^2\right), & 4x_1 x_2 & \dots \\ 4x_1 x_2 & , 2\left(x_1^2 + x_3^2 + \dots + x_n^2\right) & \dots \\ \dots & \dots & \dots \end{vmatrix}.$$

Or, dividing the ith row by $2x_i$ and the kth column by $2x_k$,

$$H(u) = (2^n x_1 x_2 \dots x_n)^2 \begin{vmatrix} \dfrac{x_2^2 + x_3^2 + \dots + x_n^2}{2x_1^2}, & 1 & \dots \\ 1 & , \dfrac{x_1^2 + x_3^2 + \dots + x_n^2}{2x_2^2} & \dots \\ \dots & \dots & \dots \end{vmatrix}.$$

This is a determinant of the form of that in IV. 25. If we write

$$3\sigma = x_1^2 + x_2^2 + \dots + x_n^2$$

$$v = (\sigma - x_1^2)(\sigma - x_2^2)\dots(\sigma - x_n^2)$$

$$H(u) = 6^n v \left\{1 + \tfrac{2}{3}\Sigma \frac{x_i^2}{\sigma - x_i^2}\right\}.$$

If $$u = x^2 y^2 + y^2 z^2 + z^2 x^2,$$

this gives

$$H(u) = 24\left\{9x^2 y^2 z^2 - (x^2 + y^2 + z^2)\, u\right\}.$$

16. Jacobians and Hessians belong to the class of functions known as covariants. That is to say, if these functions are

transformed by means of a linear substitution, the Jacobian of the transformed functions is equal to the Jacobian of the original functions multiplied by the modulus of the substitution, and the Hessian of the transformed function equal to that of the original function multiplied by the square of the modulus.

Let the variables be transformed by the substitution

$$x_i = a_{i1}\xi_1 + a_{i2}\xi_2 + \dots + a_{in}\xi_n \qquad (i = 1,\ 2 \dots n),$$

and let the functions $y_1 \dots y_n$ of $x_1 \dots x_n$ become in consequence the functions $y_1',\ y_2' \dots y_n'$ of $\xi_1 \dots \xi_n$. Since

$$\begin{aligned} \frac{dy_i'}{d\xi_k} &= \frac{dy_i}{dx_1}\frac{dx_1}{d\xi_k} + \frac{dy_i}{dx_2}\frac{dx_2}{d\xi_k} + \dots + \frac{dy_i}{dx_n}\frac{dx_n}{d\xi_k} \\ &= \frac{dy_i}{dx_1}a_{1k} + \dots + \frac{dy_i}{dx_n}a_{nk}, \end{aligned}$$

it follows from the multiplication theorem that

$$\frac{d(y_1' \dots y_n')}{d(\xi_1 \dots \xi_n)} = \frac{d(y_1 \dots y_n)}{d(x_1 \dots x_n)}|a_{ik}|,$$

which proves the theorem for Jacobians.

To prove the theorem for Hessians, let u be the original and u' the transformed function. Then since the Hessian of u is the Jacobian of $\frac{du}{dx_1} \dots \frac{du}{dx_n}$ we have

$$\begin{aligned} H(u') &= \frac{d\left(\frac{du'}{d\xi_1},\ \frac{du'}{d\xi_2} \dots \frac{du'}{d\xi_n}\right)}{d(\xi_1,\ \xi_2 \dots \xi_n)} \\ &= \frac{d\left(\frac{du'}{d\xi_1} \dots \frac{du'}{d\xi_n}\right)}{d(x_1 \dots x_n)}|a_n|. \end{aligned}$$

Now

$$\frac{d^2u'}{dx_i d\xi_k} = \frac{d^2u}{d\xi_k dx_i};$$

$$\begin{aligned} \therefore\ H(u') &= \frac{d\left(\frac{du}{dx_1} \dots \frac{du}{dx_n}\right)}{d(\xi_1 \dots \xi_n)}|a_{ik}| \\ &= \frac{d\left(\frac{du}{dx_1} \dots \frac{du}{dx_n}\right)}{d(x_1 \dots x_n)}|a_{ik}|^2 \\ &= H(u)\,.\,|a_{ik}|^2. \end{aligned}$$

17. If we have n linear functions

$$y_i = b_{i1}x_1 + \dots + b_{in}x_n \qquad (i = 1,\ 2 \dots n),$$

clearly
$$\frac{d\,(y_1 \dots y_n)}{d\,(x_1 \dots x_n)} = |b_{ik}|.$$

If u is a quadric function

$$u = b_{11}x_1^2 + \dots + 2b_{ik}x_ix_k + \dots,$$

then
$$H\,(u) = 2^n\,|b_{ik}|, \quad (b_{ik} = b_{ki}).$$

The symmetrical determinant on the right, which is called the discriminant of the quadric, is therefore an invariant which on transformation is multiplied by the square of the modulus.

CHAPTER XIV.

APPLICATIONS TO BILINEAR AND QUADRATIC FORMS.

1. A BILINEAR form is an expression which is linear and homogeneous in each of two sets of independent variables. If the number of variables in each set is n, any such form is defined by an equation

$$A = \Sigma a_{ik} x_i y_k. \qquad (i,\ k = 1,\ 2,\ \ldots\ n)$$

If B is another bilinear form with coefficients b_{ik}, a third form C can be derived from A and B, with coefficients c_{ik} which are the elements of the matrix $(a_{nn})(b_{nn})$. Thus

$$C = \Sigma a_{il} b_{lk} x_i y_k \qquad (l = 1,\ 2,\ \ldots\ n)$$
$$= \Sigma \frac{\partial A}{\partial y_l} \frac{\partial B}{\partial x_l}.$$

It is convenient to write symbolically

$$C = AB;$$

it will be observed that AB is, in general, different from BA, so that the multiplication is not commutative. But it is associative and distributive; thus, for instance, if P, Q, R denote any three forms,

$$P(Q+R) = PQ + PR, \quad P \,.\, QR = PQ \,.\, R.$$

In the particular case when $AB = BA$, the forms A, B are said to be commutable. We have a series of forms represented symbolically by positive integral powers of A; these are commutable, and obey the ordinary laws of indices.

2. With any form A are associated the matrix and the determinant of which the coefficients of the form are elements; and we have

$$(AB) = (A)\ (B)$$

with a corresponding theorem for the determinants.

The form PAQ may be derived from A by a linear transformation of each set of variables. For if we put

$$\xi_i = \frac{\partial P}{\partial y_i} = \Sigma p_{li} x_l, \quad \eta_k = \frac{\partial Q}{\partial x_k} = \Sigma q_{km} y_m$$

the form $A(\xi, \eta)$ becomes

$$\Sigma a_{ik} \frac{\partial P}{\partial y_i} \frac{\partial Q}{\partial x_k} = \Sigma \frac{\partial P}{\partial y_i} \frac{\partial}{\partial x_i} \left(\Sigma \frac{\partial A}{\partial y_k} \frac{\partial Q}{\partial x_k} \right)$$
$$= P \,.\, AQ = PAQ.$$

If the substitution is cogredient, $p_{li} = q_{il}$, and the matrices of P, Q are conjugate (v. 1): in this case we shall write $Q = P'$ and call P' the conjugate of P.

3. The form

$$E = \Sigma x_i y_i \qquad (i = 1, 2, \dots n)$$

is called the unit form. If A is an ordinary form, that is to say if $|A|$ does not vanish, there is a form A^{-1} such that

$$AA^{-1} = A^{-1}A = E.$$

This is proved by assuming $A^{-1} = \Sigma p_{ik} x_i y_k$ and equating coefficients. Clearly

$$|A|\, p_{ik} = \alpha_{ik}$$

where α_{ik} is the coefficient of a_{ik} in $|A|$. The form A^{-1} is called the reciprocal of A; its reciprocal is A itself, and if we adopt the convention that $A^0 = E$, the laws of indices hold for all integral powers of A.

Again

$$(ABC)^{-1} = C^{-1} B^{-1} A^{-1}$$

and similarly for any number of factors, if all the forms involved are ordinary.

If AB vanishes identically, either $|A| = 0$ or $|B| = 0$; in particular, if A is ordinary, $B = 0$, because in this case

$$B = A^{-1} . AB = 0;$$

hence the ordinary theory of equations may be applied to symbolical polynomials involving powers and products of ordinary commutable forms. For instance,

$$A^2 - E = A^2 - E^2 = (A + E)(A - E),$$

and if either of the factors on the right hand is ordinary, the other must vanish identically, if $A^2 = E$.

4. A form A in which $a_{ik} = 0$ except when $i = k$ may be called a normal form. Supposing that the coefficients belong to a field with the properties stated in VII. 2, it follows from VII. 10—12 and Art. 2 of the present chapter that rational unitary forms P, Q can be found such that

$$PAQ = N$$

where N is a normal form. The number of terms in N is equal to the rank of $|A|$; we shall call this the rank of A.

The determinant of A and its elementary factors are invariants of A. If A, B are any two forms, and λ an indeterminate, the result of equating to zero the determinant of $\lambda B - A$ is an equation in λ, the roots of which are invariant for simultaneous transformations of A and B. The most important case is when B is the unit form; putting

$$|\lambda E - A| = \phi(\lambda),$$

$\phi(\lambda)$ is called the characteristic function of A, and

$$\phi(\lambda) = 0$$

the characteristic equation of A.

5. Any rational function of a variable t can be reduced to the shape

$$f(t) = \frac{g(t)}{h(t)}$$

where $g(t)$, $h(t)$ are polynomials. If, now, A is any bilinear form,

$g(A)$ and $h(A)$ are the symbolical expressions of two forms derived from it. If $h(A)$ is ordinary, the form reciprocal to it will exist, and we may write

$$f(A) = g(A)[h(A)]^{-1} = g(A)/h(A).$$

Let $g(t)$ be a polynomial in t, with roots $t_1, t_2, \dots t_m$; and let

$$\phi(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n),$$

$\phi(\lambda)$ being the characteristic function of A. Then if

$$g(t) = c(t - t_1) \dots (t - t_m),$$

$$g(A) = c(A - t_1 E) \dots (A - t_m E),$$

and hence

$$\begin{aligned} |g(A)| &= c^n |A - t_1 E| \dots |A - t_m E| \\ &= (-1)^{mn} c^n \phi(t_1) \phi(t_2) \dots \phi(t_m) = (-1)^{mn} c^n \Pi (t_h - \lambda_k) \\ &= g(\lambda_1) g(\lambda_2) \dots g(\lambda_n). \end{aligned}$$

Similarly, if $f(A)$ is a rational function of A of which the denominator is ordinary,

$$|f(A)| = f(\lambda_1) f(\lambda_2) \dots f(\lambda_n).$$

Changing $f(A)$ into $\lambda E - f(A)$, which is also a rational function of A,

$$|\lambda E - f(A)| = \Pi \{\lambda - f(\lambda_i)\};$$

hence the roots of the characteristic function of $f(A)$ are $f(\lambda_1)$, $f(\lambda_2), \dots f(\lambda_n)$ where $\lambda_1, \lambda_2, \dots \lambda_n$ are the roots of the characteristic equation of A. As a particular case, the form $\phi(A)$ has a characteristic function λ^n, and the roots of its characteristic equation are all zero. As we shall presently see, the reason of this is that $\phi(A)$ vanishes identically.

6. The coefficients of A^2, A^3, etc. are rational integral functions of the n^2 coefficients a_{ik}; hence it must be possible to find $c_0, c_1, \dots c_p$, rational integral functions of the coefficients of A, such that

$$\psi(A) = c_0 A^0 + c_1 A^1 + c_2 A^2 + \dots + c_p A^p = 0$$

identically, for some value of p which does not exceed n^2. We may suppose that $\psi(A) = 0$ is the equation of lowest degree which is satisfied by A: thus c_p does not vanish.

Consider the equation

$$S = \frac{A^0}{r} + \frac{A^1}{r^2} + \frac{A^2}{r^3} + \dots$$

where the symbolical expression on the right is an infinite series, and r is an ordinary numerical quantity. By taking r large enough, the form S is interpretable, because its coefficients are convergent series in r. Multiplying by $\psi(r)$, we obtain

$$\psi(r)\, S = G(r)$$

where $G(r)$ is an integral function of r with coefficients which are integral functions of A; all the negative powers of r disappearing, in virtue of $\psi(A) = 0$. Thus S can be represented as an integral function of A with coefficients which are rational functions of r.

Again,

$$rES = A^0 + \frac{A^1}{r} + \frac{A^2}{r^2} + \dots$$

$$= A^0 + AS;$$

consequently $$(rE - A)\, S = A^0 = E$$

and $$S = (rE - A)^{-1} = \frac{F(r)}{\phi(r)},$$

where ϕ is the characteristic function of A, and

$$F(r) = \begin{vmatrix} 0, & x_1, & x_2, & \dots & x_n \\ y_1, & a_{11} - r, & a_{12}, & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ y_n, & a_{n1}, & a_{n2}, & \dots & a_{nn} - r \end{vmatrix} = \Sigma f_{ik} x_i y_k,$$

f_{ik} being a first minor of

$$|\, A - rE \,|.$$

Since

$$\frac{F(r)}{\phi(r)} = \frac{G(r)}{\psi(r)}$$

and $\psi(A)$ is the lowest function of A which vanishes, $\psi(r)$ must be prime to $G(r)$ for arbitrary values of the variables x_i, y_i and, in addition, must be a factor of $\phi(r)$. Suppose $\phi(r) = \psi(r)\, \chi(r)$; then

$$\phi(A) = \psi(A)\, \chi(A) = 0,$$

and consequently A satisfies the equation

$$\phi(A) = 0$$

where $\phi(\lambda)$ is the characteristic function of $\boldsymbol{A}$.

Evidently $\chi(r)$ is the greatest common measure of $\phi(r)$ and the quantities f_{ik}; in other words (VII. 3) it is the determinant factor D_{n-1} of $\phi(r)$ written in its determinant form. Hence also

$$\psi(r) = \phi(r)/\chi(r) = E_n,$$

the nth elementary factor of $\phi(r)$.

7. If A is an ordinary form, the constant term in $\psi(A)$ is different from zero; for otherwise, we could multiply the equation $\psi(A) = 0$ by A^{-1}, and obtain an equation of lower degree satisfied by A. Suppose, now, x being an indeterminate, that

$$\psi(x) = q(x-a)^\alpha (x-b)^\beta (x-c)^\gamma \ldots,$$

$a, b, c \ldots$ being all different from zero. We shall prove that there is an integral function of x, say $\chi(x)$, such that $\{\chi(x)\}^2 - x$ is divisible by $\psi(x)$.

Taking $\sqrt{a}$ with a determinate sign, we may write

$$\begin{aligned}\sqrt{x} &= \sqrt{\{a + (x-a)\}} \\ &= \sqrt{a}\left\{1 + \frac{(x-a)}{2a} - \frac{(x-a)^2}{8a^2} + \ldots\right\} \\ &= F(x) + (x-a)^\alpha R(x),\end{aligned}$$

where $F(x)$ is a polynomial of degree $(\alpha - 1)$, and $R(x)$ is a series proceeding by powers of $(x-a)$, which is finite when $x - a = 0$. Similarly

$$\sqrt{x} = G(x) + (x-b)^\beta S(x) = H(x) + (x-c)^\gamma T(x) = \ldots$$

and so on.

Now let

$$\chi(x) = \frac{F(x)\psi(x)}{(x-a)^\alpha} + \frac{G(x)\psi(x)}{(x-b)^\beta} + \ldots;$$

this is an integral function of x, and

$$\frac{\chi(x) - \sqrt{x}}{(x-a)^\alpha} = R(x) + \frac{G(x)}{(x-b)^\beta} + \ldots,$$

where the right-hand member is finite when $x = a$. Similarly

$$\frac{\chi(x) - \sqrt{x}}{(x-b)^\beta}, \quad \frac{\chi(x) - \sqrt{x}}{(x-c)^\gamma}, \ldots$$

are finite for $x = b, c, \ldots$ respectively. Hence

$$\frac{\{\chi(x)\}^2 - x}{\psi(x)} = \frac{\chi(x) + \sqrt{x}}{q} \cdot \frac{\chi(x) - \sqrt{x}}{(x-a)^\alpha (x-b)^\beta \ldots}$$

is finite for all finite values of x, and is therefore an integral function.

Since $\psi(A) = 0$, it follows that

$$\{\chi(A)\}^2 = A$$

and we may write

$$\chi(A) = U = A^{\frac{1}{2}}.$$

The form U is ordinary, and we may also write

$$U^{-1} = A^{-\frac{1}{2}}.$$

8. Let A, B be any two ordinary forms, and let

$$P = B(AB)^{-\frac{1}{2}} = \Sigma p_{ik} x_i y_k;$$

then

$$PAP = B(AB)^{-\frac{1}{2}} AB (AB)^{-\frac{1}{2}} = B,$$

$$P^{-1} BP^{-1} = A.$$

We have therefore found a substitution

$$x_i = \sum_k p_{ki} \xi_k, \quad y_i = \sum_k p_{ik} \eta_k$$

which converts $A(x, y)$ into $B(\xi, \eta)$. The coefficients of the substitution are rational in the square roots of the roots of the characteristic function of AB. In this field of rationality, then, any two ordinary forms are equivalent.

Let us now inquire whether A can be transformed into B by a cogredient substitution; that is, whether a form Q can be found such that

$$Q'AQ = B,$$

Q', as usual, being the conjugate of Q. We shall begin by supposing that A, B are both symmetrical, or else both skew-symmetrical.

Suppose P determined, as above, so that

$$PAP = B;$$

then since $A' = \pm A$, $B' = \pm B$, corresponding signs being taken, it follows that

$$P'AP' = B;$$

and hence that

$$(P^{-1}P')\,A\,(P'P^{-1}) = A.$$

If now we put

$$P^{-1}P' = U$$

$U' = P(P')^{-1}$, and

$$UA = AU'.$$

Hence, also,

$$U^2A = UAU' = AU'^2,$$

and, generally, if $\chi(U)$ is a polynomial in U

$$\chi(U)\,A = A\chi(U').$$

Let $\chi(U) = U^{\frac{1}{2}}$, and let

$$Q = (U')^{\frac{1}{2}}\,P';$$

then

$$Q'AQ = PU^{\frac{1}{2}}A\,(U')^{\frac{1}{2}}P' = PAU'P' = PAP = B;$$

so that two symmetrical, or two skew-symmetrical forms, if both ordinary, can always be changed one into the other by a cogredient transformation, the field of rationality being suitably extended.

More generally, if we start with

$$PAQ = B$$

and write

$$(Q')^{-1}P = U,$$

then

$$\chi(U)\,A = A\chi(U');$$

and if $\chi(U)$ is ordinary

$$P\chi(U)^{-1}A\chi(U')\,Q = B.$$

To make this a cogredient transformation, we must have

$$P\chi(U)^{-1} = Q'\chi(U),$$

or

$$\chi(U)^2 = (Q')^{-1}P = U.$$

Conversely, if

$$R = (U')^{\frac{1}{2}} Q = (P'Q^{-1})^{\frac{1}{2}} Q$$

it follows that $$R'AR = B.$$

9. Now let A, B be any two forms, A', B' their conjugates, and suppose that, u and v being indeterminates, two forms, P, Q, independent of u and v, exist, such that

$$P\,(uA + vA')\,Q = uB + vB'.$$

Write

$$A + A' = A_1, \quad A - A' = A_2,$$

$$B + B' = B_1, \quad B - B' = B_2;$$

then A_1, B_1 are both symmetrical, while A_2, B_2 are both skew-symmetrical; and moreover

$$PA_1Q = B_1, \quad PA_2Q = B_2.$$

Hence, by the method of last article, a form R can be found such that

$$R'A_1R = B_1, \quad R'A_2R = B_2,$$

and hence also

$$R'AR = \tfrac{1}{2}R'\,(A_1 + A_2)\,R = B,$$

$$R'A'R = \tfrac{1}{2}R'\,(A_1 - A_2)\,R = B'.$$

Therefore the necessary and sufficient condition that A may be transformable into B by a cogredient substitution is that forms P, Q can be found such that

$$PAQ = B, \quad PA'Q = B'.$$

This result is due to Kronecker: the proof here given is that of Frobenius. The equivalence of bilinear forms, whether ordinary or singular, has been completely discussed by Kronecker and Weierstrass; the subject is too extensive to be pursued here. It should be observed that all the theorems of this and the preceding articles of this chapter admit of a three-fold interpretation, according as we refer them to bilinear forms, determinants, or matrices. The product of two matrices has already been defined: we may call the matrix associated with the form $A + B$

the sum of the matrices associated with A and B, and then the symbolic calculus of forms becomes a calculus of matrices. From this point of view, the subject was initiated by Cayley.

10. We have already considered (VII. 10) the reduction of a matrix to a normal form; the process there given also supplies a method for the reduction of a bilinear form.

Suppose, in particular, that the form A with which we start is symmetrical. Then, in carrying out the process of reduction, we may arrange it so that the elementary transformations are made in successions of conjugate pairs. After each such pair the transformed matrix is again symmetrical, so that finally we get a cogredient transformation

$$PAP' = N = \sum_1^r e_i x_i y_i$$

where r is the rank of the matrix of A.

Let us now make the two sets of variables coincide; then A becomes a quadratic form, of which $|a_{nn}|$ is the discriminant, and we have the theorem that by a linear transformation of the variables a quadratic form can be reduced to the shape

$$e_1 x_1^2 + e_2 x_2^2 + \ldots + e_r x_r^2$$

where r is the rank of the discriminant. The discriminant is an invariant for any linear transformation, because

$$|PAP'| = |P|^2 |A|.$$

11. The reduction of a quadratic form to a sum of squares may be effected by a transformation which is rational in the coefficients of the form and in a certain number of indeterminates. The possibility of this arises from the fact that the general linear transformation involves n^2 independent coefficients, while the conditions that the new form may be a sum of squares are $\frac{1}{2}n(n-1)$ in number, and this is less than n^2.

Let us write

$$u = \Sigma a_{ik} x_i x_k, \quad u_i = \frac{du}{dx_i} = \sum_k a_{ik} x_k; \qquad (i, k = 1, 2, \ldots n)$$

and let us suppose that

$$(y_{nn})$$

is a matrix, with arbitrary elements y_{ik}. Consider the symmetrical determinant

$$U_n = \begin{vmatrix} a_{11} & a_{12} \ldots a_{1n} & y_{11} & y_{12} \ldots y_{1n} & u_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \ldots a_{nn} & y_{n1} & y_{n2} \ldots y_{nn} & u_n \\ y_{11} & y_{21} \ldots y_{n1} & 0 & 0 \ldots 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ y_{1n} & y_{2n} \ldots y_{nn} & 0 & 0 \ldots 0 & 0 \\ u_1 & u_2 \ldots u_n & 0 & 0 \ldots 0 & 0 \end{vmatrix};$$

this vanishes identically, because every minor obtained from the last $(n+1)$ rows is zero.

The value of the minor obtained by omitting the last row and column is $(-1)^n \,|\, y_{nn} \,|^2$; we denote this by R_n, and suppose that it does not vanish.

By omitting from U_n the rows and columns which contain elements y_{ik} with i or k greater than p, we obtain a symmetrical minor which we shall call U_p. The determinant obtained by omitting the last row and column of U_p we shall call R_p. Finally let X_p be the determinant derived from U_p by omitting the last column, and the last row but one. Thus X_p is a linear function of $u_1, u_2, \ldots u_n$, and therefore of $x_1, x_2, \ldots x_n$. We shall suppose, in the first instance, that $|a_{nn}|$ is different from zero.

By the properties of first minors (VI. 5)

$$R_p U_{p-1} - X_p^2 = U_p R_{p-1},$$

and hence

$$\frac{U_{p-1}}{R_{p-1}} - \frac{U_p}{R_p} = \frac{X_p^2}{R_{p-1} R_p} \quad \text{.....................(1).}$$

Now $U_n = 0$, and $U_0 = -\,|\, a_{nn} \,|\, u$, as we see by multiplying the first, second, ... nth columns by $x_1, x_2, \ldots x_n$ and subtracting from the last column. Summing the equations of which (1) is a type, from $p = 0$ to $p = n$, and writing $R_0 = |\, a_{nn} \,|$ for uniformity, we obtain

$$u = -\frac{X_1^2}{R_0 R_1} - \frac{X_2^2}{R_1 R_2} - \ldots - \frac{X_n^2}{R_{n-1} R_n} \quad \text{........... (2),}$$

which gives the required expression of u as a sum of squares.

12. At first sight it seems that a transformation has been obtained involving n^2 independent parameters; whereas we have seen that this is impossible, the number of independent constants, in general, being $\frac{1}{2}n(n+1)$. The explanation is that the constants y_{ik} enter into the quantities X_i, R_i in particular combinations, namely certain minors of $|y_{nn}|$; these are connected by identical relations, and the number of really independent parameters is reduced accordingly.

Thus, to take the simplest illustration, let

$$n = ax^2 + 2hxy + by^2, \quad (y_{22}) = \begin{pmatrix} \lambda & \mu \\ \nu & \rho \end{pmatrix};$$

then the direct application of the above process gives

$$X_1 = (\lambda\rho - \mu\nu)\{(\nu a - \lambda h)x + (\nu h - \lambda b)y\},$$

$$X_2 = -(ab - h^2)(\lambda x + \nu y),$$

$$R_0 = ab - h^2, \quad R_1 = -b\lambda^2 + 2h\lambda\mu - a\mu^2, \quad R_2 = (\lambda\rho - \mu\nu)^2;$$

and formula (2) of last article reduces to

$$ax^2 + 2hxy + by^2 = \frac{(ab - h^2)(\lambda x + \nu y)^2 + \{(a\nu - h\lambda)x + (h\nu - b\lambda)y\}^2}{b\lambda^2 - 2h\lambda\nu + a\nu^2}.$$

Here there is only one independent parameter, namely λ/ν; but instead of X_1, X_2 we may take arbitrary multiples of them, and this gives two more parameters. The method which has been explained is due to Darboux; he has shewn that it does, in fact, give the most general substitution of the kind required.

13. Let us now suppose that $|a_{nn}|$ is of rank r; that is to say, let one at least of its minors of order r be different from zero, while all those of higher order vanish. Then, in the notation of Art. 11,

$$R_0, R_1, \dots R_{n-r-1}$$

vanish identically, but R_{n-r} does not, so long as the quantities y_{ik} remain arbitrary. We shall still have

$$R_{i+1}U_i - X_i^2 = U_{i+1}R_i$$

for $i = n-r, n-r+1, \dots n$; and we conclude as before that

$$\frac{U_{n-r}}{R_{n-r}} = \frac{X^2_{n-r+1}}{R_{n-r}R_{n-r+1}} + \dots + \frac{X_n^2}{R_{n-1}R_n}.$$

In the determinant U_{n-r} subtract from the last row the sum of the products of the first n rows by $x_1, x_2, \ldots x_n$ respectively; the last row now becomes

$$0, 0, \ldots 0, -l_1, -l_2, \ldots -l_{n-r}, -u,$$

where
$$l_i = \sum_s y_{si} x_s.$$

From the last column of this new determinant subtract the sum of the products of the first n columns by $x_1, x_2, \ldots x_n$ respectively; the new determinant is symmetrical, and its value is

$$U_{n-r} = -uR_{n-r} + \Sigma l_i l_k \rho_{ik},$$

where ρ_{ik} is a minor of R_{n-r} obtained by cancelling the row and column containing y_{ik}. But $\rho_{ik}=0$, because it can be expressed as a linear function of minors of $|a_{nn}|$ which are of order higher than r: hence

$$U_{n-r} = -uR_{n-r},$$

and

$$u = -\frac{X^2_{n-r+1}}{R_{n-r}R_{n-r+1}} - \frac{X^2_{n-r+2}}{R_{n-r+1}R_{n-r+2}} - \ldots - \frac{X_n{}^2}{R_{n-1}R_n}.$$

Thus in the general case, where r, the rank of $|a_{nn}|$, is unrestricted, Darboux's method gives the reduction of u into the sum of r squares. The values given to the parameters y_{ik} must be such that none of the quantities $R_{n-r}, R_{n-r+1}, \ldots R_n$ vanishes; this is always possible, since they are functions of the quantities y_{ik} which are not identically zero.

14. Returning now to the case when $|a_{nn}|$ does not vanish, we will shew that if

$$\xi_1, \xi_2, \ldots \xi_n$$

are any assigned independent linear functions of $x_1, x_2, \ldots x_n$, we can, by a suitable choice of the quantities y_{ik}, express the quadratic in the form

$$\begin{aligned} u &= \pm (a_1\xi_1 + a_2\xi_2 + \ldots + a_n\xi_n)^2 \\ &\quad \pm (b_2\xi_2 + \ldots + b_{n-1}\xi_{n-1} + b_n\xi_n)^2 \pm \ldots \pm l_n{}^2\xi_n{}^2 \\ &= \pm \Xi_1{}^2 \pm \Xi_2{}^2 \pm \ldots \pm \Xi_n{}^2, \end{aligned}$$

where Ξ_s is a linear function of $\xi_s, \xi_{s+1}, \ldots \xi_n$.

Suppose $$\xi_k = \Sigma c_{kj} x_j, \qquad (j = 1, 2, \dots n)$$
then $|c_{nn}|$ does not vanish, and the matrix (β_{nn}) can be determined by making
$$u_i = \sum_k \beta_{ik} \xi_k \qquad (i, k = 1, 2, \dots n)$$
identically. In fact, this gives
$$(a_{nn}) = (\beta_{nn})(c_{nn}),$$
and therefore $$(\beta_{nn}) = (a_{nn})(c_{nn})^{-1}.$$

In the expressions we have denoted by R_p, U_p put $y_{ik} = \beta_{ik}$; and to fix the ideas suppose that X_p has been obtained from U_p by omitting the last row and the last column but one. Let Z_p be the determinant obtained from X_p by putting $y_{ik} = \beta_{ik}$; then if the columns headed by β_{11}, β_{12}, ... $\beta_{1, p-1}$ are multiplied by ξ_1, ξ_2, ... ξ_{p-1} respectively, and the sum of the products subtracted from the last column, the new elements of the last column, read from the top, are $v_1, v_2, \dots v_n$, where
$$v_i = \sum_{k=p}^{k=n} \beta_{ik} \xi_k.$$

Hence Z_p, when expressed as a linear function of $\xi_1, \xi_2, \dots \xi_n$, does not involve $\xi_1, \xi_2, \dots \xi_{p-1}$; and this proves the proposition.

15. The advantage of this transformation is that if we suppose
$$\xi_{p+1} = \xi_{p+2} = \dots = \xi_n = 0,$$
so that $$Z_{p+1} = Z_{p+2} = \dots = Z_n = 0,$$
the resulting values of $Z_1, Z_2, \dots Z_p$ are linearly independent. Hence we get the reduced expression of u as a sum of p squares when the variables are subject to $(n-p)$ linear relations. This reduction is important in problems of relative maxima and minima.

There are two special cases which deserve attention. The first is when $\xi_k = u_k$, so that $\beta_{ik} = \delta_{ik}$, where δ_{ik}, as usual, is Kronecker's symbol. The value of R_p is now $(-1)^p \Delta_p$, where Δ_p is the minor obtained from $|a_{nn}|$ by omitting the first p rows and columns: thus, writing Δ for $|a_{nn}|$, we have (Art. 13)
$$u = \frac{Z_1^2}{\Delta\Delta_1} + \frac{Z_2^2}{\Delta_1\Delta_2} + \dots + \frac{Z_n^2}{\Delta_{n-1}},$$
where Z_s is a linear function of $u_s, u_{s+1}, \dots u_n$.

The second case is when $\xi_k = x_k$, so that $\beta_{ik} = a_{ik}$. The value of R_p is now $(-1)^p \Delta\Delta'_p$, where Δ'_p is obtained from Δ by omitting the $(p+1)$th, ... nth rows and columns: thus, with $\Delta'_n = \Delta$ for symmetry,

$$\Delta^2 u = \frac{Z_1^2}{\Delta'_1} + \frac{Z_2^2}{\Delta'_1\Delta'_2} + \dots + \frac{Z_n^2}{\Delta'_{n-1}\Delta'_n},$$

where Z_s is now a linear function of $x_s, x_{s+1}, \dots x_n$.

Sylvester has proved that when a quadric is linearly transformed to a sum of squares by a real substitution, the number of positive and negative squares is always the same. The results of the present article shew that the variations of sign are determined by either of the sets

$$\Delta\Delta_1,\ \Delta_1\Delta_2,\ \dots\ \Delta_{n-2}\Delta_{n-1},\ \Delta_{n-1},$$
$$\Delta'_1,\ \Delta'_1\Delta'_2,\ \dots\ \Delta'_{n-2}\Delta'_{n-1},\ \Delta'_{n-1}\Delta'_n.$$

In particular, the necessary and sufficient conditions that all the squares may be positive are that either of the series

$$\Delta,\ \Delta_1,\ \dots\ \Delta_{n-1},$$
$$\Delta'_1,\ \Delta'_2,\ \dots\ \Delta'_n$$

should consist of terms which are all positive. In the case when the variables are subject to $(n-p)$ linear relations, the variations of sign in the reduced form of u are obtained from

$$\begin{vmatrix} (a_{nn}) & (\beta_{np}) \\ (\beta_{np})' & (0_{pp}) \end{vmatrix}$$

and its leading minors, the elements β_{ik} being determined as in Art. 14.

16. If a quadric, by means of a linear transformation, has been reduced to the sum of n squares,

$$\begin{aligned} u &= \Sigma a_{ik} x_i x_k \\ &= A_1 y_1^2 + A_2 y_2^2 + \dots + A_n y_n^2; \end{aligned}$$

the discriminant of the right-hand side is $A_1 A_2 \dots A_n$, and hence if μ is the modulus of transformation,

$$A_1 A_2 \dots A_n = \mu^2 \,|\, a_{nn} \,|.$$

Two given quadrics

$$u = \Sigma a_{ik} x_i x_k, \quad v = \Sigma b_{ik} x_i x_k$$

can, in general, by a simultaneous linear transformation

$$x_i = c_{i1}y_1 + c_{i2}y_2 + \dots + c_{in}y_n \qquad (i = 1, 2 \dots n)$$

be reduced, each to the sum of n squares of the same linear functions, viz.

$$u = A_1 y_1^2 + A_2 y_2^2 + \dots + A_n y_n^2,$$
$$v = s_1 A_1 y_1^2 + s_2 A_2 y_2^2 + \dots + s_n A_n y_n^2;$$

for in order to determine the n^2 constants, c_{ik}, we have first $n(n-1)$ equations from the fact that the coefficients of the products $y_i y_k$ must vanish, and n additional equations from the condition that the ratio of the coefficients of y_i^2 is to be s_i; in all, n^2 equations.

If we form the discriminant of $su - v$, its value for the original quadrics is

$$| sa_{nn} - b_{nn} | \dots\dots\dots\dots\dots\dots(1),$$

and for the transformed quadrics

$$A_1 \dots A_n (s - s_1)(s - s_2) \dots (s - s_n) \dots\dots\dots\dots(2).$$

The ratio of the quantities (1) and (2) is μ^2; hence $s_1 \dots s_n$ are the roots of the equation

$$\Delta(s) = | sa_{nn} - b_{nn} | = 0 \dots\dots\dots\dots(3).$$

17. The following resolution is due to Darboux.

If we write

$$F = su - v, \quad X_i = \tfrac{1}{2}\frac{dF}{dx_i} = su_i - v_i \dots\dots\dots\dots(4),$$

we have identically by Art. 11

$$F = su - v = -\frac{1}{\Delta(s)}\begin{vmatrix} sa_{11} - b_{11} & \dots & sa_{1n} - b_{1n}, & X_1 \\ \dots & \dots & \dots & \dots \\ sa_{n1} - b_{n1} & \dots & sa_{nn} - b_{nn}, & X_n \\ X_1 & \dots & X_n & \end{vmatrix} \dots(5).$$

The determinant on the right is a function of s of order $n-1$; resolve the fraction into partial fractions, and we get

$$su - v = -\Sigma\frac{1}{\Delta'(s_i)(s - s_i)}\begin{vmatrix} s_i a_{11} - b_{11} & \dots & s_i a_{1n} - b_{1n}, & X_1 \\ \dots & \dots & \dots & \dots \\ s_i a_{n1} - b_{n1} & \dots & s_i a_{nn} - b_{nn}, & X_n \\ X_1 & \dots & X_n & \end{vmatrix} \dots(6).$$

The determinants on the right are all perfect squares by VI. 6, for they are obtained by bordering the vanishing determinant $\Delta(s_i)$. Whence

$$su - v = \Sigma \frac{U_i^2}{\Delta'(s_i)(s - s_i)},$$

where U_i is a linear function of the form

$$U_i = d_{i1}X_1 + \ldots + d_{in}X_n.$$

If in the determinant (6) we replace X_i by its value from (4), and subtract from the last column the first n multiplied by $x_1 \ldots x_n$, and do the same for the rows, the value of the determinant is unaltered, but X_i is replaced by $\frac{1}{2}(s - s_i)\frac{du}{dx_i}$.

A term is also introduced in the principal diagonal in the last place, but since its minor vanishes by (3) we may replace it by zero. Thus U_i is replaced by

$$\begin{aligned} U_i' &= \tfrac{1}{2}(s - s_i)\left(d_{i1}\frac{du}{dx_1} + \ldots + d_{in}\frac{du}{dx_n}\right) \\ &= (s - s_i)V_i, \end{aligned}$$

where V_i is independent of s;

$$\therefore\ su - v = \Sigma \frac{V_i^2(s - s_i)}{\Delta'(s_i)}.$$

Equating coefficients of s we get

$$u = \Sigma \frac{V_i^2}{\Delta'(s_i)}, \quad v = \Sigma \frac{s_i V_i^2}{\Delta'(s_i)},$$

which is the required resolution.

It is assumed here that $s_1, s_2, \ldots s_n$ are all different: when this is not the case, the analysis requires modification. For a complete discussion, the memoirs of Weierstrass, Kronecker, and Darboux should be consulted.

18. An important branch of the theory of quadrics is that of their linear automorphic transformation. That is to say, as the name implies, the discussion of those linear transformations which do not alter the outward appearance of the quadric. So that if $x_1 \ldots x_n$ are the original, and $y_1 \ldots y_n$ the new variables,

$$\Sigma a_{ik}x_i x_k \text{ becomes } \Sigma a_{ik}y_i y_k.$$

Without entering into a discussion of the general case we shall study that particular one which gave rise to the whole theory.

In the transformation from one set of rectangular axes in space to another with the same origin, the distance of a point from the origin is the same, and expressing this for the two systems

$$x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2;$$

such a transformation is linear and automorphic, and is known as an orthogonal transformation.

19. The general case of an orthogonal transformation is to determine those linear transformations which give us

$$x_1^2 + x_2^2 + \dots + x_n^2 = y_1^2 + y_2^2 + \dots + y_n^2.$$

The theory is due to Cayley, but we shall give it here as modified by Veltmann.

Let us consider the following equations

$$\left.\begin{array}{l} b_{11}x_1 + b_{12}x_2 + \dots + b_{1n}x_n = b_{11}y_1 + b_{21}y_2 + \dots + b_{n1}y_n \\ b_{21}x_1 + b_{22}x_2 + \dots + b_{2n}x_n = b_{12}y_1 + b_{22}y_2 + \dots + b_{n2}y_n \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ b_{n1}x_1 + b_{n2}x_2 + \dots + b_{nn}x_n = b_{1n}y_1 + b_{2n}y_2 + \dots + b_{nn}y_n \end{array}\right\} \dots (1),$$

where the system b_{ik} is skew, so that

$$b_{ik} = -b_{ki}, \quad b_{ii} = z \dots\dots\dots\dots\dots (2).$$

The rows of coefficients on the right coincide with the columns on the left.

Let $$B = | b_{ik} | = | b_{ki} |,$$

so that B is a skew determinant, and let B_{ik} be the system of first minors. Solving the system of equations (1) we get

$$y_i = c_{i1}x_1 + c_{i2}x_2 + \dots + c_{in}x_n$$

$$x_k = d_{k1}y_1 + d_{k2}y_2 + \dots + d_{kn}y_n.$$

The coefficient of x_k in y_i is given by

$$Bc_{ik} = B_{i1}b_{1k} + B_{i2}b_{2k} + \dots + B_{in}b_{nk}.$$

If $$s = B_{i1}b_{k1} + B_{i2}b_{k2} + \dots + B_{in}b_{kn},$$

then $$Bc_{ik} + s = 2B_{ik}b_{kk}.$$

Now $s = B$ or 0 according as i is or is not equal to k, thus

$$c_{ik} = \frac{2B_{ik}z}{B}, \quad c_{ii} = \frac{2B_{ii}z - B}{B}.$$

In the same way

$$d_{ki} = \frac{2B_{ik}z}{B}, \quad d_{ii} = \frac{2B_{ii}z - B}{B}.$$

Thus

$$c_{ik} = d_{ki},$$

and we may write

$$y_i = c_{i1}x_1 + c_{i2}x_2 + \dots + c_{in}x_n$$
$$x_i = c_{1i}y_1 + c_{2i}y_2 + \dots + c_{ni}y_n.$$

Substitute for $x_1 \dots x_n$ from the second of these systems in the first and equate coefficients of y_k and y_i on both sides, thus

$$\left.\begin{aligned} c_{i1}^2 + c_{i2}^2 + \dots + \quad c_{in}^2 &= 1 \\ c_{i1}c_{k1} + c_{i2}c_{k2} + \dots + c_{in}c_{kn} &= 0 \end{aligned}\right\}.$$

If we substitute from the first system in the second, we get

$$\left.\begin{aligned} c_{1i}^2 + c_{2i}^2 + \dots + \quad c_{ni}^2 &= 1 \\ c_{1i}c_{1k} + c_{2i}c_{2k} + \dots + c_{ni}c_{nk} &= 0 \end{aligned}\right\}.$$

Whence we see at once that

$$x_1^2 + x_2^2 + \dots + x_n^2 = y_1^2 + y_2^2 + \dots + y_n^2,$$

and thus the coefficients c_{ik} are those of an orthogonal substitution.

20. By the preceding article we are able to express the n^2 coefficients of an orthogonal transformation by means of the $\frac{1}{2}n(n-1)$ quantities

$$\begin{array}{l} b_{12},\ b_{13} \dots b_{1n} \\ \qquad b_{23} \dots b_{2n} \\ \qquad \dots\dots\dots\dots \\ \qquad\qquad\quad b_{n-1n}, \end{array}$$

by forming a skew determinant with these, the elements of whose leading diagonal are equal to z; and without loss of generality we may put $z = 1$.

For the case $n = 2$, let

$$B = \begin{vmatrix} 1, & \lambda \\ -\lambda, & 1 \end{vmatrix} = 1 + \lambda^2;$$

the system of first minors is

$$\begin{matrix} 1, & \lambda \\ -\lambda, & 1. \end{matrix}$$

Hence the coefficients of a binary orthogonal transformation are

$$\begin{matrix} \dfrac{1-\lambda^2}{1+\lambda^2}, & \dfrac{2\lambda}{1+\lambda^2}, \\[2ex] \dfrac{-2\lambda}{1+\lambda^2}, & \dfrac{1-\lambda^2}{1+\lambda^2}. \end{matrix}$$

For a ternary orthogonal transformation

$$B = \begin{vmatrix} 1, & \nu, & -\mu \\ -\nu, & 1, & \lambda \\ \mu, & -\lambda, & 1 \end{vmatrix} = 1 + \lambda^2 + \mu^2 + \nu^2;$$

the system of first minors is

$$\begin{matrix} 1+\lambda^2, & \nu + \lambda\mu, & -\mu + \lambda\nu, \\ -\nu + \lambda\mu, & 1 + \mu^2, & \lambda + \mu\nu, \\ \mu + \lambda\nu, & -\lambda + \mu\nu, & 1 + \nu^2. \end{matrix}$$

Hence the coefficients of the ternary orthogonal transformation are

$$\begin{matrix} \dfrac{1+\lambda^2-\mu^2-\nu^2}{B}, & 2\dfrac{\nu+\lambda\mu}{B}, & 2\dfrac{-\mu+\lambda\nu}{B}, \\[2ex] 2\dfrac{-\nu+\lambda\mu}{B}, & \dfrac{1+\mu^2-\lambda^2-\nu^2}{B}, & 2\dfrac{\lambda+\mu\nu}{B}, \\[2ex] 2\dfrac{\mu+\lambda\nu}{B}, & 2\dfrac{-\lambda+\mu\nu}{B}, & \dfrac{1+\nu^2-\lambda^2-\mu^2}{B}. \end{matrix}$$

If we write

$$\lambda = \cos f \tan \tfrac{1}{2}\theta, \quad \mu = \cos g \tan \tfrac{1}{2}\theta, \quad \nu = \cos h \tan \tfrac{1}{2}\theta,$$

where $$\cos^2 f + \cos^2 g + \cos^2 h = 1,$$

and therefore $$B = \sec^2 \tfrac{1}{2}\theta,$$

we get Rodrigues' formulæ.

For the quaternary orthogonal transformation

$$B = \begin{vmatrix} 1, & a, & b, & c \\ -a, & 1, & h, & -g \\ -b, & -h, & 1, & f \\ -c, & g, & -f, & 1 \end{vmatrix}.$$

Then

$$B = 1 + a^2 + b^2 + c^2 + f^2 + g^2 + h^2 + \theta^2,$$

where $$\theta = af + bg + ch.$$

And the system of first minors is

$$\begin{array}{ll} B_{11} = \quad 1 + f^2 + g^2 + h^2, & B_{12} = \quad a + f\theta - bh + cg, \\ B_{21} = -a - f\theta + cg - bh, & B_{22} = \quad 1 + f^2 + b^2 + c^2, \\ B_{31} = -b - cf - g\theta + ah, & B_{32} = -h + fg - ab - c\theta, \\ B_{41} = -c + bf - ag - h\theta, & B_{42} = \quad g + fh + b\theta - ca, \end{array}$$

$$\begin{array}{ll} B_{13} = \quad b + g\theta - cf + ah, & B_{14} = \quad c + h\theta - ag + bf, \\ B_{23} = \quad h + fg + c\theta - ab, & B_{24} = -g + hf - ac - b\theta, \\ B_{33} = \quad 1 + g^2 + c^2 + a^2, & B_{34} = \quad f + gh + a\theta - bc, \\ B_{43} = -f + gh - bc - a\theta, & B_{44} = \quad 1 + h^2 + a^2 + b^2. \end{array}$$

Thus the coefficients of the quaternary orthogonal transformation are

$$\begin{aligned} Bc_{11} &= 1 - \theta^2 + f^2 - a^2 + g^2 - b^2 + h^2 - c^2, \\ Bc_{12} &= 2\,(a + f\theta - bh + cg), \\ Bc_{13} &= 2\,(b + g\theta - cf + ah), \\ Bc_{14} &= 2\,(c + h\theta - ag + bf), \\ &\text{\&c.} \end{aligned}$$

21. The square of the determinant of an orthogonal substitution is unity, for

$$|\, c_{ik} \,|^2 = |\, d_{ik} \,|,$$

where $$d_{ik} = c_{1i}c_{1k} + c_{2i}c_{2k} + \ldots + c_{ni}c_{nk},$$

i.e. $$d_{ik} = 0, \quad d_{ii} = 1;$$

$$\therefore \quad |\, c_{ik} \,|^2 = 1, \text{ or } |\, c_{ik} \,| = \epsilon,$$

where ϵ means ± 1.

22. If C_{ik} is the complement of c_{ik} in C, then

$$C_{ik} = \epsilon \,.\, c_{ik}.$$

For we have the system of equations

$$\begin{gathered} c_{11}c_{1k} + \dots + c_{n1}c_{nk} = 0 \\ \dots\dots\dots\dots\dots\dots \\ c_{1k}c_{1k} + \dots + c_{nk}c_{nk} = 1 \\ \dots\dots\dots\dots\dots\dots \\ c_{1n}c_{1k} + \dots + c_{nn}c_{nk} = 0. \end{gathered}$$

Multiply these equations by $C_{i1}, C_{i2} \dots C_{in}$ and add, the coefficient of c_{ik} is ϵ, the others vanish; thus

$$C_{ik} = \epsilon c_{ik}.$$

23. Any minor of the system c_{ik} is equal to its complementary minor.

For
$$\begin{vmatrix} C_{11} \dots C_{1p} \\ \dots\dots\dots \\ C_{p1} \dots C_{pp} \end{vmatrix} = \epsilon^{p-1} \begin{vmatrix} c_{p+1\,p+1} \dots c_{p+1\,n} \\ \dots\dots\dots\dots \\ c_{n\,p+1} \quad \dots c_{nn} \end{vmatrix},$$
by VI. 6. But
$$\begin{vmatrix} C_{11} \dots C_{1p} \\ \dots\dots\dots \\ C_{p1} \dots C_{pp} \end{vmatrix} = \epsilon^{p} \begin{vmatrix} c_{11} \dots c_{1p} \\ \dots\dots\dots \\ c_{p1} \dots c_{pp} \end{vmatrix}$$
by the theorem just proved. Hence
$$\epsilon \begin{vmatrix} c_{11} \dots c_{1p} \\ \dots\dots\dots \\ c_{p1} \dots c_{pp} \end{vmatrix} = \begin{vmatrix} c_{p+1\,p+1} \dots c_{p+1\,n} \\ \dots\dots\dots\dots \\ c_{n\,p+1} \quad \dots c_{nn} \end{vmatrix}.$$

24. If $A^{(n)} = | a_{nn} |$, $B^{(n)} = | b_{nn} |$ be two determinants of orthogonal substitutions of order n, then the determinant
$$P(\lambda, \mu) = | \lambda a_{nn} + \mu b_{nn} |$$
is not altered by interchanging λ and μ.

For the symbolical expression for $P(\lambda, \mu)$ is
$$\begin{aligned} P(\lambda, \mu) &= (\lambda A + \mu B)^n \\ &= A^{(n)} B^{(n)} \left(\frac{\lambda}{B} + \frac{\mu}{A}\right)^n, \end{aligned}$$
as in VI. 7. And as there proved
$$P(\lambda, \mu) = A^{(n)} B^{(n)} \left| \frac{\lambda B_{ik}}{B^{(n)}} + \frac{\mu A_{ik}}{A^{(n)}} \right|.$$

Or, if $A^{(n)} = 1 = B^{(n)}$, we have by Art. 22,

$$\begin{aligned} P(\lambda, \mu) &= | \lambda b_{ik} + \mu a_{ik} | \\ &= P(\mu, \lambda). \end{aligned}$$

From this we see, that if from the coefficients of an orthogonal substitution of order n we subtract the corresponding coefficients of another orthogonal substitution of the same order, the determinant formed with these differences vanishes if n is odd.

25. If we take n quadrics in n variables we may conveniently represent them by the system of equations

$$u_i = \Sigma a_{ijk} x_j x_k \qquad (i, j, k = 1, 2 \ldots n).$$

With the coefficients a_{ijk} we can form a cubic determinant of order n which will be an invariant of the system of quadrics $u_1 \ldots u_n$. Zehfuss has pointed out that for three ternary quadrics this gives Aronhold's invariant, while the auxiliary expressions he gives for its calculation are the cubic minors of the second order.

For the two binary quadrics

$$\begin{aligned} &ax^2 + 2bxy + cy^2, \\ &a'x^2 + 2b'xy + c'y^2, \end{aligned}$$

it is the harmonic invariant

$$aa' - 2bb' + cc'.$$

The general theorem is that for n quantics of order p in n variables the determinant of class $(p+1)$, which can be formed with their coefficients, is an invariant of the system. By allowing all the quantics to become identical we get an invariant of a single quantic when it is of even order.

CHAPTER XV.

DETERMINANTS OF FUNCTIONS OF THE SAME VARIABLE.

1. If $y_1, y_2 \dots y_n$ are functions of a variable x, and if

$$y_i^{(k)} = \frac{d^k y_i}{dx^k},$$

the determinant

$$\Sigma \pm y_1 y_2^{(1)} \dots y_n^{(n-1)} = \begin{vmatrix} y_1 & , & y_2 & \dots & y_n \\ y_1^{(1)} & , & y_2^{(1)} & \dots & y_n^{(1)} \\ \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & , & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$

is called the determinant of the functions $y_1, y_2 \dots y_n$, and is denoted by $D(y_1, y_2 \dots y_n)$.

2. If y is any function of x, and we multiply the above determinant by

$$\begin{vmatrix} y, & 0, & 0 & \dots 0 \\ y^{(1)}, & y, & 0 & \dots 0 \\ y^{(2)}, & 2y^{(1)}, & y & \dots 0 \\ \dots & \dots & \dots & \dots \\ y^{(n-1)}, & (n-1)_1 y^{(n-2)}, & (n-1)_2 y^{(n-3)} & \dots y \end{vmatrix} = y^n,$$

combining the columns of D with the rows of the latter, we obtain

$$D(yy_1, yy_2 \dots yy_n) = y^n D(y_1, y_2 \dots y_n).$$

In particular if we put $yy_1 = 1$ in the determinant on the left, all the elements in the first column vanish, except the first, which

is unity, and the determinant reduces to the determinant of the $n-1$ functions

$$\frac{d}{dx}\left(\frac{y_2}{y_1}\right) = \frac{D(y_1,\ y_2)}{y_1^2},\ \dots \frac{d}{dx}\left(\frac{y_n}{y_1}\right) = \frac{D(y_1,\ y_n)}{y_1^2}.$$

If therefore we put

$$D(y_1,\ y_i) = y_i',$$

then
$$D(y_1,\ y_2 \dots y_n) = \frac{1}{y_1^{n-2}} D(y_2',\ y_3' \dots y_n').$$

3. If the functions $y_1 \dots y_n$ are connected by any linear relation

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0,$$

it is plain by differentiating this $n-1$ times, and eliminating $c_1 \dots c_n$ between the original and these $n-1$ new equations, that we get:

$$D(y_1,\ y_2 \dots y_n) = 0.$$

Conversely if the determinant of the functions $y_1 \dots y_n$ vanishes, then they are connected by a linear equation with constant coefficients. We shall prove this by induction; we shall assume that if the determinant of $n-1$ functions vanishes, these functions are linearly connected, and we shall shew that the same is true for n functions. If y_1 does not vanish, which would be equivalent to a linear relation among the functions, it follows from the preceding article that since

$$D(y_1,\ y_2 \dots y_n) = 0,$$

we must also have

$$D(y_2',\ y_3' \dots y_n') = 0.$$

Hence by hypothesis the $n-1$ functions $y_2' \dots y_n'$ are linearly connected, i.e. we have

$$c_2 y_2' + c_3 y_3' + \dots + c_n y_n' = 0.$$

Dividing by y_1^2 we get

$$c_2 \frac{d}{dx}\left(\frac{y_2}{y_1}\right) + c_3 \frac{d}{dx}\left(\frac{y_3}{y_1}\right) + \dots + c_n \frac{d}{dx}\left(\frac{y_n}{y_1}\right) = 0,$$

or integrating

$$c_1 y_1 + c_2 y_2 + \dots + c_n y_n = 0.$$

Thus if the theorem is true for $n-1$ functions, it is true for n, but it is clearly true for two functions, and hence generally.

4. From the formula

$$D(y_1,\ y_2 \dots y_n) = \frac{1}{y_1^{n-2}} D(y_2',\ y_3' \dots y_n'),$$

it follows that

$$D(y_1,\ y_2,\ y_3) = \frac{1}{y_1} D(y_2',\ y_3')$$

$$D(y_1,\ y_2,\ y_4) = \frac{1}{y_1} D(y_2',\ y_4')$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$D(y_1,\ y_2,\ y_n) = \frac{1}{y_1} D(y_2',\ y_n').$$

The same formula also gives

$$D(y_2',\ y_3' \dots y_n') = \frac{1}{y_2'^{\,n-3}} D\{D(y_2',\ y_3'),\ \ D(y_2',\ y_4') \dots D(y_2',\ y_n')\}.$$

Combining these formulæ, we obtain the equation

$$D(y_1,\ y_2 \dots y_n) = \frac{1}{[D(y_1,\ y_2)]^{n-3}} D\{D(y_1,\ y_2,\ y_3),$$
$$D(y_1,\ y_2,\ y_4) \dots D(y_1,\ y_2,\ y_n)\}.$$

By repeated application of this method we obtain the theorem :—

If $u_1,\ u_2 \dots u_m,\ \ v_1,\ v_2 \dots v_n$ be functions of x, and if

$$w_i = D(u_1,\ u_2 \dots u_m,\ v_i) \qquad (i = 1,\ 2 \dots n),$$

then $$D(u_1,\ u_2 \dots u_m,\ \ v_1,\ v_2 \dots v_n) = \frac{D(w_1,\ w_2 \dots w_n)}{\{D(u_1,\ u_2 \dots u_m)\}^{n-1}}.$$

5. A special case of this theorem is

$$D(y_1 \dots y_{k-1},\ y_{k+1} \dots y_n,\ y_k,\ y)$$
$$= \frac{D\{D(y_1 \dots y_{k-1},\ y_{k+1} \dots y_n,\ y_k),\ D(y_1 \dots y_{k-1},\ y_{k+1} \dots y_n,\ y)\}}{D(y_1 \dots y_{k-1},\ y_{k+1} \dots y_n)},$$

which we may write in the form

$$\frac{D(y_1 \dots y_n,\ y)\, D(y_1 \dots y_{k-1},\ y_{k+1} \dots y_n)}{D(y_1 \dots y_n)\, D(y_1 \dots y_n)} = -\frac{d}{dx} \frac{D(y,\ y_1 \dots y_{k-1},\ y_{k+1} \dots y_n)}{D(y_1 \dots y_n)}.$$

Assuming now that the functions $y_1 \dots y_n$ are independent, let us write

$$z_k = (-1)^{n+k} \frac{D(y_1 \dots y_{k-1},\ y_{k+1} \dots y_n)}{D(y_1 \dots y_n)}$$

$$P(y) = (-1)^n \frac{D(y,\ y_1 \dots y_n)}{D(y_1 \dots y_n)}$$

$$P(y,\ z_k) = (-1)^{k-1} \frac{D(y,\ y_1 \dots y_{k-1},\ y_{k+1} \dots y_n)}{D(y_1 \dots y_n)},$$

then the above equation can be written

$$z_k P(y) = \frac{d}{dx} P(y,\ z_k).$$

6. The determinant

$$\begin{vmatrix} y_1 \ , & y_2 & \dots & y_n \\ y_1^{(1)} \ , & y_2^{(1)} & \dots & y_n^{(1)} \\ \dots & \dots & \dots & \dots \\ y_1^{(n-2)}, & y_2^{(n-2)} & \dots & y_n^{(n-2)} \\ y_1^{(k)} \ , & y_2^{(k)} & \dots & y_n^{(k)} \end{vmatrix}$$

vanishes if $k < n-1$, but if $k = n-1$ its value is $D(y_1, y_2 \dots y_n)$. Expanding it according to the elements of the last row we get the system of equations

$$\left.\begin{array}{l} y_1 z_1 + y_2 z_2 + \dots + y_n z_n = 0 \\ y_1^{(1)} z_1 + y_2^{(1)} z_2 + \dots + y_n^{(1)} z_n = 0 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_1^{(n-2)} z_1 + y_2^{(n-2)} z_2 + \dots + y_n^{(n-2)} z_n = 0 \\ y_1^{(n-1)} z_1 + y_2^{(n-1)} z_2 + \dots + y_n^{(n-1)} z_n = 1 \end{array}\right\} \dots\dots(A).$$

If we write $s_{pq} = y_1^{(p)} z_1^{(q)} + y_2^{(p)} z_2^{(q)} + \dots + y_n^{(p)} z_n^{(q)}$, we can write these more briefly

$$s_{00} = 0,\quad s_{10} = 0 \dots s_{n-2,0} = 0,\quad s_{n-1,0} = 1.$$

Now we have

$$\frac{ds_{k-1,0}}{dx} = s_{k,0} + s_{k-1,1}$$

$$\frac{d^2 s_{k-2,0}}{dx^2} = s_{k,0} + 2s_{k-1,1} + s_{k-2,2}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{d^k s_{00}}{dx^k} = s_{k,0} + ks_{k-1,1} + k_2 s_{k-2,2} + \dots + s_{0,k}.$$

If $k < n-1$, it follows from these equations that

$$s_{\alpha\beta} = 0, \text{ if } \alpha + \beta < n-1.$$

If $k = n-1$, it follows since

$$(1-1)^r = 1 - r + r_2 - \ldots + (-1)^r = 0$$

that $s_{n-1,0}$, $s_{n-2,1} \ldots$ are alternately equal to $+1$ and -1.

Hence we get the following theorem: The expression s_{pq} is equal to zero when $p+q < n-1$, and equal to $(-1)^q$ when $p+q = n-1$.

If $k = n$, we conclude in the same way that

$$s_{n,0} = -s_{n-1,1} = s_{n-2,2} = \ldots = (-1)^n s_{0,n}.$$

7. Among the relations just established we have

$$\left.\begin{array}{l} z_1 y_1 + z_2 y_2 + \ldots + z_n y_n = 0 \\ z_1^{(1)} y_1 + z_2^{(1)} y_2 + \ldots + z_n^{(1)} y_n = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ z_1^{(n-2)} y_1 + z_2^{(n-2)} y_2 + \ldots + z_n^{(n-2)} y_n = 0 \\ z_1^{(n-1)} y_1 + z_2^{(n-1)} y_2 + \ldots + z_n^{(n-1)} y_n = (-1)^{n-1} \end{array}\right\} \ldots\ldots(\text{B}).$$

If $D(z_1, z_2 \ldots z_n)$ vanished, it would follow, since

$$s_{00} = 0, \; s_{01} = 0 \ldots s_{0,n-2} = 0,$$

that $s_{0,n-1}$ would also vanish, while its value is $(-1)^{n-1}$. Thus the functions $z_1, z_2 \ldots z_n$ are not linearly connected with each other. Comparing the systems (A) and (B) it appears that the relation between $y_1 \ldots y_n$ and $z_1 \ldots z_n$ is a reciprocal one, if we neglect the sign when n is even. From each relation between these systems we deduce a new one by interchanging

$$y_1, \; y_2 \ldots y_n, \; z_1, \; z_2 \ldots z_n$$

with $\quad (-1)^{n-1} z_1, \; (-1)^{n-1} z_2 \ldots (-1)^{n-1} z_n, \; y_1, \; y_2 \ldots y_n.$

Thus from the equation

$$z_k = (-1)^{n+k} \frac{D(y_1 \ldots y_{k-1}, \; y_{k+1} \ldots y_n)}{D(y_1, \; y_2 \ldots y_n)}$$

we deduce

$$y_k = (-1)^{k-1} \frac{D(z_1 \ldots z_{k-1}, \; z_{k+1} \ldots z_n)}{D(z_1, \; z_2 \ldots z_n)}.$$

In consequence of this we shall call $z_1 \ldots z_n$ the conjugates of $y_1 \ldots y_n$.

8. If we form the product by rows of the two following determinants

$$\begin{vmatrix} y_1 & \dots & y_k, & y_{k+1} & \dots & y_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_1^{(k-1)} & \dots & y_k^{(k-1)}, & y_{k+1}^{(k-1)} & \dots & y_n^{(k-1)} \\ y_1^{(k)} & \dots & y_k^{(k)}, & y_{k+1}^{(k)} & \dots & y_n^{(k)} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & y_k^{(n-1)}, & y_{k+1}^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix},$$

$$\begin{vmatrix} 1 & \dots & 0, & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 1, & 0 & \dots & 0 \\ z_1 & \dots & z_k, & z_{k+1} & \dots & z_n \\ \dots & \dots & \dots & \dots & \dots & \dots \\ z_1^{(n-k-1)} & \dots & z_k^{(n-k-1)}, & z_{k+1}^{(n-k-1)} & \dots & z_n^{(n-k-1)} \end{vmatrix},$$

the first of which is $D(y_1 \dots y_n)$, the second $D(z_{k+1} \dots z_n)$, we get

$$\begin{vmatrix} y_1 & \dots & y_k, & s_{00} & \dots & s_{0,\,n-k-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_1^{(k-1)} & \dots & y_k^{(k-1)}, & s_{k-1,\,0} & \dots & s_{k-1,\,n-k-1} \\ y_1^{(k)} & \dots & y_k^{(k)}, & s_{k,\,0} & \dots & s_{k,\,n-k-1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y_1^{(n-1)} & \dots & y_k^{(n-1)}, & s_{n-1,\,0} & \dots & s_{n-1,\,n-k-1} \end{vmatrix}.$$

In this determinant the elements common to the first k rows and last $n-k$ columns all vanish, whence it reduces to

$$\begin{vmatrix} y_1 & \dots & y_k \\ \dots & \dots & \dots \\ y_1^{(k-1)} & \dots & y_k^{(k-1)} \end{vmatrix} \begin{vmatrix} s_{k,\,0} & \dots & s_{k,\,n-k-1} \\ \dots & \dots & \dots \\ s_{n-1,\,0} & \dots & s_{n-1,\,n-k-1} \end{vmatrix}.$$

The first of these $= D(y_1 \dots y_k)$; in the second all the elements to the left of the second diagonal vanish, whence its value is

$$(-1)^{\frac{(n-k)(n-k+1)}{2}} s_{n-1,\,0}\, s_{n-2,\,1} \dots s_{k,\,n-k-1}$$
$$= 1.$$

Thus we have

$$D(y_1 \dots y_n)\, D(z_{k+1} \dots z_n) = D(y_1 \dots y_k).$$

If $k = 0$, we have

$$D(y_1 \dots y_n)\, D(z_1 \dots z_n) = 1.$$

9. From this last equation we get

$$P(y) = (-1)^n \frac{D(y, y_1 \dots y_n)}{D(y_1, y_2 \dots y_n)}$$

$$= (-1)^n (y, y_1 \dots y_n)\, D(z_1, z_2 \dots z_n).$$

Or

$$P(y) = (-1)^n \begin{vmatrix} y, & y_1 & \dots & y_n \\ y^{(1)}, & y_1^{(1)} & \dots & y_n^{(1)} \\ \dots & \dots & \dots & \dots \\ y^{(n)}, & y_1^{(n)} & \dots & y_n^{(n)} \end{vmatrix} \begin{vmatrix} 1, & 0 & \dots & 0 \\ 0, & z_1 & \dots & z_n \\ \dots & \dots & \dots & \dots \\ 0, & z_1^{(n-1)} & \dots & z_n^{(n-1)} \end{vmatrix}$$

$$= (-1)^n \begin{vmatrix} y, & s_{00}, & s_{01} & \dots & s_{0,n-1} \\ y^{(1)}, & s_{10}, & s_{11} & \dots & s_{1,n-1} \\ \dots & \dots & \dots & \dots & \dots \\ y^{(n)}, & s_{n0}, & s_{n1} & \dots & s_{n,n-1} \end{vmatrix}.$$

Similarly we should get

$$P(z) = (-1)^n \begin{vmatrix} z, & z^{(1)} & \dots & z^{(n)} \\ s_{00}, & s_{01} & \dots & s_{0n} \\ \dots & \dots & \dots & \dots \\ s_{n-1,0}, & s_{n-1,1} & \dots & s_{n-1,n} \end{vmatrix}.$$

10. These determinants occur in the theory of linear differential equations. Thus, consider the equation

$$a_0 y + a_1 y^{(1)} + \dots + a_n y^{(n)} = 0$$

where the quantities $a_0, a_1 \dots a_n$ do not contain y. If $y_1 \dots y_n$ are n particular integrals, we have the n equations

$$a_0 y_i + a_1 y_i^{(1)} + \dots + a_n y_i^{(n)} = 0 \qquad (i = 1, 2 \dots n),$$

and by eliminating the a's we get

$$\begin{vmatrix} y, & y^{(1)} & \dots & y^{(n)} \\ y_1, & y_1^{(1)} & \dots & y_1^{(n)} \\ \dots & \dots & \dots & \dots \\ y_n, & y_n^{(1)} & \dots & y_n^{(n)} \end{vmatrix} = 0,$$

or

$$D(y, y_1 \dots y_n) = 0.$$

If we solve the equations for $\frac{a_{n-1}}{a_n}$ we get

$$\begin{vmatrix} y_1, & y_1^{(1)} \dots y_1^{(n-2)}, & y_1^{(n)} \\ \dots & \dots & \dots \\ y_n, & y_n^{(1)} \dots y_n^{(n-2)}, & y_n^{(n)} \end{vmatrix} \div \begin{vmatrix} y_1, & y_1^{(1)} \dots y_1^{(n-1)} \\ \dots & \dots \\ y_n, & y_n^{(1)} \dots y_n^{(n-1)} \end{vmatrix} = -\frac{a_{n-1}}{a_n},$$

i.e.
$$\frac{d}{dx} \log D(y_1, y_2 \dots y_n) = -\frac{a_{n-1}}{a_n},$$

$$D(y_1, y_2 \dots y_n) = \exp. \left(-\int \frac{a_{n-1}}{a_n}\, dx\right).$$

11. Though not immediately connected with the subject of the present chapter we shall give Hesse's solution of Jacobi's differential equation.

This equation is

$$-A_1 d\eta + A_2 d\xi + A_3(\xi d\eta - \eta d\xi) = 0,$$

where
$$A_i = a_{i1}\xi + a_{i2}\eta + a_{i3} \qquad (i = 1, 2, 3).$$

We can write the equation in the form of the determinant

$$\begin{vmatrix} \xi, & \eta, & 1 \\ d\xi, & d\eta, & 0 \\ A_1, & A_2, & A_3 \end{vmatrix} = 0.$$

Now let $\xi = \frac{x}{z}$, $\eta = \frac{y}{z}$; the equation becomes

$$\begin{vmatrix} x, & y, & z \\ zx' - z'x, & zy' - yz', & 0 \\ A_1, & A_2, & A_3 \end{vmatrix} = 0.$$

Multiply the first row by z' and add it to the second, this divides by z, and we get

$$\begin{vmatrix} x, & y, & z \\ x', & y', & z' \\ A_1, & A_2, & A_3 \end{vmatrix} = 0.$$

Now let us multiply this equation by

$$\begin{vmatrix} \alpha_1, & \beta_1, & \gamma_1 \\ \alpha_2, & \beta_2, & \gamma_2 \\ \alpha_3, & \beta_3, & \gamma_3 \end{vmatrix},$$

and let
$$p_i = \alpha_i x + \beta_i y + \gamma_i z.$$

Also assume that

$$\lambda_i p_i = A_1\alpha_i + A_2\beta_i + A_3\gamma_i.$$

Then

$$\begin{vmatrix} p_1, & p_2, & p_3 \\ dp_1, & dp_2, & dp_3 \\ \lambda_1 p_1, & \lambda_2 p_2, & \lambda_3 p_3 \end{vmatrix} = 0,$$

i.e.

$$\begin{vmatrix} \dfrac{dp_1}{p_1}, & \dfrac{dp_2}{p_2}, & \dfrac{dp_3}{p_3} \\ 1, & 1, & 1 \\ \lambda_1, & \lambda_2, & \lambda_3 \end{vmatrix} = 0, \qquad \text{or} \quad \begin{vmatrix} \log p_1, & \log p_2, & \log p_3 \\ 1, & 1, & 1 \\ \lambda_1, & \lambda_2, & \lambda_3 \end{vmatrix} = C,$$

or, as we may write it

$$p_1^{\lambda_2-\lambda_3} . p_2^{\lambda_3-\lambda_1} . p_3^{\lambda_1-\lambda_2} = C.$$

Since we assumed that

$$A_1\alpha_i + A_2\beta_i + A_3\gamma_i = \lambda_i p_i,$$

we have, by equating coefficients of x, y, z

$$\alpha_1(a_{11}-\lambda) + \beta_1 a_{12} + \gamma_1 a_{13} = 0,$$
$$\alpha_1 a_{21} + \beta_1(a_{22}-\lambda) + \gamma_1 a_{23} = 0,$$
$$\alpha_1 a_{31} + \beta_1 a_{32} + \gamma_1(a_{33}-\lambda) = 0.$$

Hence eliminating α_1, β_1, γ_1, we see that λ_1, λ_2, λ_3 are the roots of the equation

$$\begin{vmatrix} a_{11}-\lambda, & a_{12}, & a_{13} \\ a_{21}, & a_{22}-\lambda, & a_{23} \\ a_{31}, & a_{32}, & a_{33}-\lambda \end{vmatrix} = 0.$$

CHAPTER XVI.

APPLICATIONS TO THE THEORY OF CONTINUED FRACTIONS.

1. THE application of the theory of determinants to continued fractions gives great facility in the discussion of these functions. As usual in English mathematical works we shall denote the continued fraction

$$\cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{a_3 + \cdots + \cfrac{b_n}{a_n}}}}$$

by

$$\frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +}\,\frac{b_3}{a_3 +}\cdots\frac{b_n}{+\,a_n}.$$

Such a fraction is called a descending continued fraction.

In addition to these we shall discuss a less known form of continued fractions, which, however, is historically the older form of the two, namely, the ascending continued fraction

$$\cfrac{b_1 + \cfrac{b_2 + \cdots}{a_2}}{a_1},$$

which, in an analogous manner, will be denoted by

$$\frac{b_1 +}{a_1}\,\frac{b_2 +}{a_2}\cdots\frac{+\,b_n}{a_n}.$$

Our object is to establish a determinant expression for the convergents to these two forms.

2. If we write down the system of equations

$$\begin{aligned} b_1 x &= a_1 x_1 + x_2 \\ b_2 x_1 &= a_2 x_2 + x_3 \\ b_3 x_2 &= a_3 x_3 + x_4 \\ &\cdots\cdots\cdots\cdots \end{aligned}$$

we see that

$$\frac{x_1}{x} = \frac{b_1}{a_1 + \dfrac{x_2}{x_1}}, \quad \frac{x_2}{x_1} = \frac{b_2}{a_2 + \dfrac{x_3}{x_2}} \cdots$$

Hence $\dfrac{x_1}{x}$ is the continued fraction

$$\frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +} \cdots$$

3. If we are to determine the nth convergent, i.e. the value of the fraction when we stop at $\dfrac{b_n}{a_n}$, we must suppose that x_{n+1} and all succeeding x's vanish, whence we have the system of equations

$$\begin{aligned} b_1 x &= a_1 x_1 + x_2 \\ 0 &= -b_2 x_1 + a_2 x_2 + x_3 \\ 0 &= - b_3 x_2 + a_3 x_3 + x_4 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 &= - b_n x_{n-1} + a_n x_n. \end{aligned}$$

Solving this set of equations for x_1 we get:

$$\begin{vmatrix} a_1, & 1, & 0 & \dots & & \\ -b_2, & a_2, & 1 & \dots & & \\ 0, & -b_3, & a_3 & \dots & & \\ \dots & \dots & \dots & & & \\ 0, & 0, & 0 & \dots & a_{n-1}, & 1 \\ 0, & 0, & 0 & \dots & -b_n, & a_n \end{vmatrix} x_1 = \begin{vmatrix} b_1 x, & 1, & 0 & \dots & \\ 0, & a_2, & 1 & \dots & \\ 0, & -b_3, & a_3 & \dots & \\ \dots & \dots & \dots & & \\ 0, & 0, & 0 & \dots -b_n, & a_n \end{vmatrix}.$$

Thus

$$\frac{x_1}{x} = b_1 \begin{vmatrix} a_2, & 1 & \dots & 0, & 0 \\ -b_3, & a_3 & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0 & \dots & a_{n-1}, & 0 \\ 0, & 0 & \dots & -b_n, & a_n \end{vmatrix} \div \begin{vmatrix} a_1, & 1, & 0 & \dots & 0, & 0 \\ -b_2, & a_2, & 1 & \dots & 0, & 0 \\ 0, & -b_3, & a_3 & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 & \dots & a_{n-1}, & 1 \\ 0, & 0, & 0 & \dots & -b_n, & a_n \end{vmatrix}$$

or

$$\frac{x_1}{x} = \frac{p_n}{q_n}$$

say, where

$$p_n = b_1 \begin{vmatrix} a_2, & 1, & 0, & 0 & \ldots & 0, & 0 \\ -b_3, & a_3, & 1, & 0 & \ldots & 0, & 0 \\ 0, & -b_4, & a_4, & 1 & \ldots & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & 0 & \ldots & a_{n-1}, & 1 \\ 0, & 0, & 0, & 0 & \ldots & -b_n, & a_n \end{vmatrix}$$

$$= a_n p_{n-1} + b_n p_{n-2},$$

if we expand (IV. 24) according to the elements of the last row and column.

Similarly

$$q_n = \begin{vmatrix} a_1, & 1, & 0, & 0 & \ldots & 0, & 0 \\ -b_2, & a_2, & 1, & 0 & \ldots & 0, & 0 \\ 0, & -b_3, & a_3, & 1 & \ldots & 0, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0, & 0, & 0, & 0 & \ldots & a_{n-1}, & 1 \\ 0, & 0, & 0, & 0 & \ldots & -b_n, & a_n \end{vmatrix}$$

$$= a_n q_{n-1} + b_n q_{n-2}.$$

Since $p_n = b_1 \dfrac{dq_n}{da_1}$, we can write the convergent in the form

$$b_1 \frac{d}{da_1} (\log q_n).$$

4. The determinants of the form q_n have been called continuants by Mr Muir. Since

$$q_n = a_n q_{n-1} + b_n q_{n-2},$$

if u_n is the number of terms in the continuant of order n

$$u_n = u_{n-1} + u_{n-2},$$

an equation of differences which gives

$$u_n = A \left(\frac{1+\sqrt{5}}{2}\right)^n + B \left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Since $u_1 = 1$, $u_2 = 2$, we have

$$u_n = \{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}\} \div 2^{n+1}\sqrt{5}.$$

It is easy to shew by the binomial theorem that this number is an integer. Prof. Sylvester obtains this number in the form of the series

$$1+(n-1)+\frac{(n-2)(n-3)}{1.2}+\frac{(n-3)(n-4)(n-5)}{1.2.3}+\dots$$

5. The value of the continuant q_n is the same as that of the determinant

$$q_n' = \begin{vmatrix} a_1, & c_1, & 0 & \dots & 0 \\ d_2, & a_2, & c_2 & \dots & 0 \\ 0\,, & d_3, & a_3 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0\,, & 0\,, & 0 & \dots & a_n \end{vmatrix},$$

provided only

$$c_r d_{r+1} = -\,b_{r+1} \qquad (r=1,\ 2\dots n-1).$$

This is clear if we expand by IV. 24, according to the elements which stand in the last row and column. For then

$$\begin{aligned} q_n' &= a_n q'_{n-1} - d_n c_{n-1} q'_{n-2} \\ &= a_n q'_{n-1} + b_n q'_{n-2}, \end{aligned}$$

where $q_1' = q_1$, $q_2' = q_2$. Hence $q_n' = q_n$, the equation of differences being linear.

Thus we can also write

$$q_n = \begin{vmatrix} a_1, & -1, & 0\,, & 0 & \dots \\ b_2, & a_2\,, & -1, & 0 & \dots \\ 0\,, & b_3\,, & a_3\,, & -1 & \dots \\ 0\,, & 0\,, & b_4\,, & a_4 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix}.$$

6. The value of the continued fraction is not altered if we replace

$$b_r,\quad a_r,\quad b_{r+1}$$

by

$$kb_r,\ ka_r,\ kb_{r+1}.$$

For the quotient $\frac{p_n}{q_n}$ is unaltered if we multiply numerator and denominator by any the same number. If we multiply both by k, the row

$$\dots -b_r,\ a_r,\ 1\dots$$

in the determinants equated to p_n and q_n in Art. 3 is replaced by

$$\ldots - kb_r,\ ka_r,\ k \ldots$$

and by Art. 5, in place of the last k, we can write unity if we replace b_{r+1} by kb_{r+1}.

Since then we can write the continued fraction

$$\frac{b_1}{a_1 +}\,\frac{b_2}{a_2 +} \cdots \frac{b_n}{+ a_n}$$

in the form

$$\frac{b_1 \dfrac{k}{a_1}}{k +}\;\frac{b_2 \dfrac{k^2}{a_1 a_2}}{k +}\;\frac{b_3 \dfrac{k^2}{a_2 a_3}}{k +} \cdots \;+\; \frac{b_n \dfrac{k^2}{a_{n-1} a_n}}{k};$$

q_n can be written in the form of the skew determinant

$$\begin{vmatrix} k\ , & \alpha_1\ , & 0\ , & 0 & \ldots \\ -\alpha_1, & k\ , & \alpha_2\ , & 0 & \ldots \\ 0\ , & -\alpha_2, & k\ , & \alpha_3 & \ldots \\ 0\ , & 0\ , & -\alpha_3, & k & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

where
$$\alpha_r = \sqrt{\left(\frac{b_{r+1} k^2}{a_r a_{r+1}}\right)}.$$

Thus the convergents to a continued fraction can always be represented by the quotient of two skew determinants.

7. In any determinant D we have

$$D \frac{d^2 D}{da_{11}\, da_{nn}} = \frac{dD}{da_{11}}\,\frac{dD}{da_{nn}} - \frac{dD}{da_{1n}}\,\frac{dD}{da_{n1}}.$$

For D take the continuant q_n (Art. 5), then

$$\frac{d^2 D}{da_{11}\, da_{nn}} = \frac{1}{b_1} \cdot p_{n-1}, \qquad \frac{dD}{da_{nn}} = q_{n-1}, \qquad \frac{dD}{da_{11}} = \frac{1}{b_1} p_n,$$

$$\frac{dD}{da_{1n}} = b_2 b_3 \ldots b_n, \qquad \frac{dD}{da_{n1}} = (-1)^{n-1}.$$

Thus
$$q_n p_{n-1} - q_{n-1} p_n = (-1)^n\, b_1 b_2 \ldots b_n.$$

8. In the case of the ascending continued fraction

$$\frac{b_1 +}{a_1}\,\frac{b_2 +}{a_2} \cdots$$

it is clear that if the nth convergent be $\frac{p_n}{q_n}$, the scale of relation is

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + b_n}{a_n q_{n-1}}.$$

Hence $$q_n = a_1 a_2 \ldots a_n.$$

To determine p_n we have the system of equations:

$$\begin{aligned}
p_1 \qquad\qquad\qquad\qquad\qquad\qquad\qquad &= b_1\\
-a_2p_1 + p_2 \qquad\qquad\qquad\qquad\qquad &= b_2\\
-a_3p_2 + p_3 \qquad\qquad\qquad\qquad &= b_3\\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots &\\
-a_{n-1}p_{n-2} + p_{n-1} \qquad &= b_{n-1}\\
-a_n p_{n-1} + p_n &= b_n.
\end{aligned}$$

The determinant of this system is unity, all the elements to the right of the leading diagonal vanishing;

$$\therefore\ p_n = \begin{vmatrix} 1, & 0, & 0 \ldots & 0, & b_1\\ -a_2, & 1, & 0 \ldots & 0, & b_2\\ 0, & -a_3, & 1 \ldots & 0, & b_3\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ 0, & 0, & 0 \ldots & 1, & b_{n-1}\\ 0, & 0, & 0 \ldots & -a_n, & b_n \end{vmatrix}.$$

Multiply all the columns except the last by -1, and move the last column to the first place; the determinant is unchanged, thus

$$p_n = \begin{vmatrix} b_1, & -1, & 0 \ldots & 0, & 0\\ b_2, & a_2, & -1 \ldots & 0, & 0\\ b_3, & 0, & a_3 \ldots & 0, & 0\\ \ldots & \ldots & \ldots & \ldots & \ldots\\ b_n, & 0, & 0 \ldots & 0, & a_n \end{vmatrix}.$$

The nth convergent to the fraction is

$$\frac{p_n}{a_1 a_2 \ldots a_n}.$$

The number of terms in p_n is n.

9. By means of these determinant expressions for the convergents we can transform an ascending continued fraction into a descending continued fraction.

In the determinant p_n of the preceding article multiply the rth row, beginning with the last, by b_{r-1}, and subtract from it the $(r-1)$st row multiplied by b_r, and do this for all the rows. The determinant is multiplied by the factor

$$b_1 b_2 \dots b_{n-1} = k^{-1},$$

say, and

$$p_n = k \begin{vmatrix} b_1, & -1, & 0 & \dots & & & \\ 0, & a_2 b_1 + b_2, & -b_1 & \dots & & & \\ 0, & -a_2 b_3, & a_3 b_2 + b_3 & \dots & & & \\ \dots & \dots & \dots & \dots & & & \\ 0, & 0, & 0 & \dots & a_{n-2} b_{n-3} + b_{n-2}, & -b_{n-3}, & 0 \\ 0, & 0, & 0 & \dots & -a_{n-2} b_{n-1}, & a_{n-1} b_{n-2} + b_{n-1}, & -b_{n-2} \\ 0, & 0, & 0 & \dots & 0, & -a_{n-1} b_n, & a_n b_{n-1} + b_n \end{vmatrix}.$$

Similarly, since

$$q_n = a_1 a_2 \dots a_n$$
$$= \begin{vmatrix} a_1, & -1, & 0 & \dots & 0, & 0 \\ 0, & a_2, & -1 & \dots & 0, & 0 \\ 0, & 0, & a_3 & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 & \dots & a_{n-1}, & -1 \\ 0, & 0, & 0 & \dots & 0, & a_n \end{vmatrix},$$

$$q_n = k \begin{vmatrix} a_1, & -1, & 0 & \dots & \\ -a_1 b_2, & a_2 b_1 + b_2, & -b_1 & \dots & \\ 0, & -a_2 b_3, & a_3 b_2 + b_3 & \dots & \\ \dots & \dots & \dots & \dots & \\ 0, & 0, & 0 & \dots -a_{n-1} b_n, & a_n b_{n-1} + b_n \end{vmatrix}.$$

Now on inspection it is clear that these determinants are continuants, as defined in Art. 3, whose 2nd, 3rd ... $(n-1)$st rows have been multiplied by $b_1, b_2 \dots b_{n-2}$ respectively; also

$$p_n = b_1 \frac{dq_n}{da_1}.$$

Hence by Arts. 3 and 6

$$\frac{p_n}{q_n} = \frac{b_1}{a_1 -} \frac{a_1 b_2}{a_2 b_1 + b_2 -} \frac{a_2 b_1 b_3}{a_3 b_2 + b_3} \cdots \frac{a_{n-2} b_{n-3} b_{n-1}}{a_{n-1} b_{n-2} + b_{n-1} -} \frac{a_{n-1} b_{n-2} b_n}{a_n b_{n-1} + b_n},$$

which gives us a rule for transforming an ascending continued fraction into a descending continued fraction, the number of quotients in each being the same.

10. We can make immediate use of this theorem to deduce a formula of Euler's, by means of which a series can be converted into a continued fraction.

Take the series

$$S = A_1 - A_2 + A_3 - A_4 + \ldots + (-1)^{n-1} A_n$$

$$= \begin{vmatrix} A_1, & 1, & 0, & 0 \ldots 0 \\ A_2, & 1, & 1, & 0 \ldots 0 \\ A_3, & 0, & 1, & 1 \ldots 0 \\ \cdots & \cdots & \cdots & \cdots \\ A_n, & 0, & 0, & 0 \ldots 1 \end{vmatrix},$$

as we see by subtracting from each row the one below it, beginning with the last, when the determinant reduces to its principal term. Multiplying each column after the first by -1, we reduce the determinant to the continuant for an ascending continued fraction. Thus the above series is equal to:

$$(-1)^{n-1} \frac{A_1 +}{1} \frac{A_2 +}{-1} \cdots \frac{A_{n-1} +}{-1} \frac{A_n}{-1},$$

and transforming this by the rule just obtained to a descending continued fraction

$$S = (-1)^{n-1} \frac{A_1}{1 -} \frac{A_2}{A_2 - A_1 +} \frac{A_1 A_3}{A_3 - A_2 +} \cdots \frac{A_{n-2} A_n}{A_n - A_{n-1}}$$

$$= \frac{A_1}{1 +} \frac{A_2}{A_1 - A_2 +} \frac{A_1 A_3}{A_2 - A_3 +} \cdots \frac{A_{n-2} A_n}{A_{n-1} - A_n}.$$

If the original series is

$$s = \frac{1}{A_1} - \frac{1}{A_2} + \frac{1}{A_3} \cdots$$

we can obtain its form as a continued fraction by altering the continuant to S in accordance with Art. 6, when we get

$$s = \frac{1}{A_1 +}\,\frac{A_1^2}{A_2 - A_1 +}\,\frac{A_2^2}{A_3 - A_2 +}\cdots$$

11. Various generalisations of continued fractions have been devised by Jacobi and others. The following generalisation, due to Fürstenau, is taken from a review of his memoir by Günther.

If x and y are any two real numbers, and we write

$$y = a_0 + \frac{x_1}{y_1},\quad y_1 = a_1 + \frac{x_2}{y_2},\quad y_2 = a_2 + \frac{x_3}{y_3}\cdots$$

$$x = b_0 + \frac{1}{y_1},\quad x_1 = b_1 + \frac{1}{y_2},\quad x_2 = b_2 + \frac{1}{y_3}\cdots$$

where a and b are the greatest integers contained in x and y, then on substituting we have:

$$y = a_0 + \cfrac{b_1 + \cfrac{1}{a_2 + \cfrac{b_3 + \frac{1}{a_4}}{a_3 + \frac{b_4}{a_4}}}}{a_1 + \cfrac{b_2 + \cfrac{1}{a_3 + \frac{b_4}{a_4}}}{a_2 + \cfrac{b_3 + \frac{1}{a_4}}{a_3 + \frac{b_4}{a_4}}}}$$

and

$$x = b_0 + \cfrac{1}{a_1 + \cfrac{b_2 + \cfrac{1}{a_3 + \cfrac{b_4 +}{a_4 +}}}{a_2 + \cfrac{b_3 + \cfrac{1}{a_4 +}}{a_3 + \frac{b_4}{a_4}}}}.$$

If now all that stands to the left of one of the vertical lines be called a first, second ... convergent, and if we denote the numerators of x and y by X_p, Y_p, while the denominator, which is clearly the same for both, is called N_p, we shall have

$$(Y, X, N)_{p+1} = a_{p+1}(Y, X, N)_p + b_{p+1}(Y, X, N)_{p-1} + (Y, X, N)_{p-2}.$$

Thus the equations have four instead of three terms, and we get

$$Y_p = \begin{vmatrix} a_0, & b_1, & 1, & 0 \dots 0 \\ -1, & a_1, & b_2, & 1 \dots 0 \\ 0, & -1, & a_2, & b_3 \dots 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & 0 \dots a_p \end{vmatrix},$$

$$X_p = \begin{vmatrix} b_0, & 1, & 0, & 0 \dots 0 \\ -1, & a_1, & b_2, & 1 \dots 0 \\ 0, & -1, & a_2, & b_3 \dots 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & 0 \dots a_p \end{vmatrix},$$

$$N_p = \begin{vmatrix} a_1, & b_2, & 1, & 0 \dots 0 \\ -1, & a_2, & b_3, & 1 \dots 0 \\ 0, & -1, & a_3, & b_4 \dots 0 \\ \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & 0 \dots a_p \end{vmatrix}.$$

Corresponding to the theorem of Art. 7 we have now

$$\begin{vmatrix} Y_{p+1}, & Y_p, & Y_{p-1} \\ X_{p+1}, & X_p, & X_{p-1} \\ N_{p+1}, & N_p, & N_{p-1} \end{vmatrix} = 1.$$

12. If ordinary continued fractions be called fractions of the first class, those in Art. 11 may be called fractions of the second class.

Fürstenau extends the idea still further, and summing up his results we may state them as follows: If we seek to determine n quantities $x_1, x_2 \dots x_n$ as fractions of the form

$$x_1 = \frac{X_1}{N}, \quad x_2 = \frac{X_2}{N} \dots x_n = \frac{X_n}{N}$$

each such fraction can be written as a continued fraction of the $(n-1)$th class. The pth convergents to these continued fractions take the form

$$\frac{X_{p1}}{N_p},\ \frac{X_{p2}}{N_p}\cdots\frac{X_{pn}}{N_p},$$

and if

$$\begin{array}{ll} a_{11} & \ldots\ a_{1,\,n+1} \\ a_{21} & \ldots\ a_{2,\,n+1} \\ \multicolumn{2}{c}{\dotfill} \\ a_{n+1,\,1} & \ldots\ a_{n+1,\,n+1} \end{array}$$

are the quotients entering into the continued fractions, then

$$X_{pq} = a_{1p}X_{p-1,\,q} + a_{2p}X_{p-2,\,q} + \ldots + a_{n+1,\,p}X_{p-n-1,\,q},$$
$$N_p = a_{1p}N_{p-1} + a_{2p}N_{p-2} + \ldots + a_{n+1,\,p}N_{p-n-1}.$$

The quotients X and N are always connected by the equation

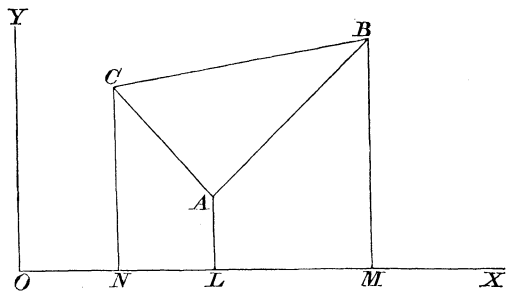
$$\begin{vmatrix} X_{p1}, & X_{p-1,\,1}, & X_{p-2,\,1} & \ldots & X_{p-n,\,1} \\ X_{p2}, & X_{p-1,\,2}, & X_{p-2,\,2} & \ldots & X_{p-n,\,2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ X_{pn}, & X_{p-1,\,n}, & X_{p-2,\,n} & \ldots & X_{p-n,\,n} \\ N_p, & N_{p-1}, & N_{p-2} & \ldots & N_{p-n} \end{vmatrix} = (-1)^{np}.$$

The author also shews that the real roots of an equation of the nth order can be represented as periodic continued fractions of the $(n-1)$th class.

CHAPTER XVII.

APPLICATIONS TO GEOMETRY.

1. THE axes being rectangular let the co-ordinates of the angular points of a triangle ABC be (x_1, y_1) (x_2, y_2) (x_3, y_3). Then if Δ is the area of the triangle it is plain from the figure that



$$\Delta = \text{trap. } BN. - \text{trap. } BL - \text{trap. } CL$$

$$= \tfrac{1}{2}(y_2 + y_3)(x_2 - x_3) - \tfrac{1}{2}(y_2 + y_1)(x_2 - x_1) - \tfrac{1}{2}(y_3 + y_1)(x_1 - x_3),$$

or
$$2\Delta = y_3x_2 - y_2x_3 + x_3y_1 - x_1y_3 + x_1y_2 - x_2y_1$$

$$= \begin{vmatrix} 1, & 1, & 1 \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix} = \begin{vmatrix} 1, & x_1, & y_1 \\ 1, & x_2, & y_2 \\ 1, & x_3, & y_3 \end{vmatrix}.$$

If the axes were oblique this would have to be multiplied by the sine of the angle between the axes. Thus

$$2\Delta = \sin(XY) \begin{vmatrix} 1, & 1, & 1 \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix},$$

where (XY) is the angle between the axes. This form, however, is not often used, and unless the fact is specially mentioned the axes are supposed to be rectangular.

If we multiply the first row by x_1 and subtract it from the second, then the first row by y_1 and subtract it from the third, we get

$$2\Delta = \begin{vmatrix} x_2 - x_1, & x_3 - x_1 \\ y_2 - y_1, & y_3 - y_1 \end{vmatrix}.$$

It must be noticed that the area of a triangle changes sign if we alter the cyclical order of the letters. Thus ABC and ACB are equal triangles, whose areas are opposite in sign; ABC and BCA are equal in magnitude and agree in sign.

2. Let the co-ordinates of the angular points of a tetrahedron $ABCD$ be $(x_1, y_1, z_1) \ldots (x_4, y_4, z_4)$. Let V be its volume.

Let Δ be the area of the triangle BCD, and let the equation of its plane be

$$(x - x_2)\cos\alpha + (y - y_2)\cos\beta + (z - z_2)\cos\gamma = 0.$$

The projection of the triangle BCD on the plane of xy is $\Delta\cos\gamma$, and the co-ordinates of its angular points are

$$(x_2, y_2)\ (x_3, y_3)\ (x_4, y_4);$$

thus, by Art. 1,

$$2\Delta\cos\gamma = \begin{vmatrix} x_3 - x_2, & x_4 - x_2 \\ y_3 - y_2, & y_4 - y_2 \end{vmatrix}.$$

Similarly we get

$$2\Delta\cos\beta = \begin{vmatrix} z_3 - z_2, & z_4 - z_2 \\ x_3 - x_2, & x_4 - x_2 \end{vmatrix}, \qquad 2\Delta\cos\alpha = \begin{vmatrix} y_3 - y_2, & y_4 - y_2 \\ z_3 - z_2, & z_4 - z_2 \end{vmatrix}.$$

If p is the perpendicular from A on the plane BCD,

$$-p = (x_1 - x_2)\cos\alpha + (y_1 - y_2)\cos\beta + (z_1 - z_2)\cos\gamma.$$

Hence

$$\begin{aligned} -6V &= -2\Delta p \\ &= 2\Delta\cos\alpha\,(x_1 - x_2) + 2\Delta\cos\beta\,(y_1 - y_2) + 2\Delta\cos\gamma\,(z_1 - z_2) \\ &= (x_1 - x_2)\begin{vmatrix} y_3 - y_2, & y_4 - y_2 \\ z_3 - z_2, & z_4 - z_2 \end{vmatrix} + (y_1 - y_2)\begin{vmatrix} z_3 - z_2, & z_4 - z_2 \\ x_3 - x_2, & x_4 - x_2 \end{vmatrix} \\ &\qquad + (z_1 - z_2)\begin{vmatrix} x_3 - x_2, & x_4 - x_2 \\ y_3 - y_2, & y_4 - y_2 \end{vmatrix} \end{aligned}$$

$$= \begin{vmatrix} x_1 - x_2, & x_3 - x_2, & x_4 - x_2 \\ y_1 - y_2, & y_3 - y_2, & y_4 - y_2 \\ z_1 - z_2, & z_3 - z_2, & z_4 - z_2 \end{vmatrix} = - \begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1 - x_2, & 0, & x_3 - x_2, & x_4 - x_2 \\ y_1 - y_2, & 0, & y_3 - y_2, & y_4 - y_2 \\ z_1 - z_2, & 0, & z_3 - z_2, & z_4 - z_2 \end{vmatrix}.$$

If in this last determinant we multiply the first row by x_2, y_2, z_2 and add it to the second, third and fourth rows respectively, we obtain

$$6V = \begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix}.$$

3. If the tetrahedron be referred to oblique axes through the same origin, and if the cosines of the angles these make with the rectangular axes be given by the scheme

	X	Y	Z
x	l_1	l_2	l_3
y	m_1	m_2	m_3
z	n_1	n_2	n_3

$$x = Xl_1 + Yl_2 + Zl_3, \text{ \&c.};$$

whence

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1, & 1 \\ X_1, & X_2, & X_3, & X_4 \\ Y_1, & Y_2, & Y_3, & Y_4 \\ Z_1, & Z_2, & Z_3, & Z_4 \end{vmatrix} \begin{vmatrix} 1, & 0, & 0, & 0 \\ 0, & l_1, & m_1, & n_1 \\ 0, & l_2, & m_2, & n_2 \\ 0, & l_3, & m_3, & n_3 \end{vmatrix}.$$

Now let

$$D = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix};$$

then remembering that

$$l_1^2 + m_1^2 + n_1^2 = 1,$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = \cos XY, \text{ \&c.},$$

we have

$$D^2 = \begin{vmatrix} 1, & \cos XY, & \cos XZ \\ \cos YX, & 1, & \cos YZ \\ \cos ZX, & \cos ZY, & 1 \end{vmatrix}.$$

This determinant is usually called the square of the sine of the solid angle contained by the oblique axis, in analogy with the determinant

$$\sin^2 XY = \begin{vmatrix} 1, & \cos XY \\ \cos YX, & 1 \end{vmatrix}$$

in a plane. Thus

$$D^2 = \sin^2 (XYZ).$$

And in oblique co-ordinates

$$6V = \begin{vmatrix} 1, & 1, & 1, & 1 \\ X_1, & X_2, & X_3, & X_4 \\ Y_1, & Y_2, & Y_3, & Y_4 \\ Z_1, & Z_2, & Z_3, & Z_4 \end{vmatrix} \sin (XYZ).$$

4. From the determinant expressions in Arts. 1 and 2 we can at once write down a number of geometrical relations.

If the distances x be measured along a straight line from a fixed point, we see that

$$\begin{vmatrix} 1, & x_i \\ 1, & x_k \end{vmatrix} = (x_k - x_i) = (ki)$$

is the distance between the two points marked k and i. The determinant

$$\begin{vmatrix} 1, & x_1, & 1, & x_1 \\ 1, & x_2, & 1, & x_2 \\ 1, & x_3, & 1, & x_3 \\ 1, & x_4, & 1, & x_4 \end{vmatrix}$$

vanishes identically, because it has several columns alike. Expanding it by IV. 5 according to products of minors from the first two and last two columns, we get

$$(12)(34) + (13)(42) + (14)(23) = 0.$$

Or, if we call the points A, B, C, D, this is the well-known relation between the segments formed by four collinear points

$$AB \,.\, CD + AC \,.\, DB + AD \,.\, BC = 0.$$

If we expand the vanishing determinant

$$\left| 1,\ x_i,\ y_i,\ 1,\ x_i,\ y_i \right| \qquad (i = 1, 2 \ldots 6)$$

according to minors from the first three and last three columns, we get no geometrical relation, the terms cancelling each other in pairs.

But if we expand the determinant

$$\left| 1,\ x_i,\ y_i,\ z_i,\ 1,\ x_i,\ y_i,\ z_i \right| = 0 \qquad (i = 1,\ 2 \ldots 8)$$

according to the products of minors from the first and last four columns we get an identical relation of thirty-five terms between the volumes of the tetrahedra formed by eight points.

5. Again, for five points,

$$\begin{vmatrix} 1, & 1, & 1, & 1, & 1 \\ 1, & 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4, & x_5 \\ y_1, & y_2, & y_3, & y_4, & y_5 \\ z_1, & z_2, & z_3, & z_4, & z_5 \end{vmatrix} = 0.$$

If v_1 = volume of tetrahedron (2345) and we expand the determinant according to the elements of the first row, by IV. 10, we get

$$v_1 + v_2 + v_3 + v_4 + v_5 = 0.$$

6. By the theorem VI. 20,

$$\begin{vmatrix} 1, & 1, & 1 \\ x_1, & x_2, & x_3 \\ y_1, & y_2, & y_3 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ \xi_1, & \xi_2, & \xi_3 \\ \eta_1, & \eta_2, & \eta_3 \end{vmatrix} = \begin{vmatrix} 1, & 1, & 1 \\ x_1, & \xi_1, & \xi_2 \\ y_1, & \eta_1, & \eta_2 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ \xi_3, & x_2, & x_3 \\ \eta_3, & y_2, & y_3 \end{vmatrix}$$
$$+ \begin{vmatrix} 1, & 1, & 1 \\ x_1, & \xi_2, & \xi_3 \\ y_1, & \eta_2, & \eta_3 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ \xi_1, & x_2, & x_3 \\ \eta_1, & y_2, & y_3 \end{vmatrix} + \begin{vmatrix} 1, & 1, & 1 \\ x_1, & \xi_3, & \xi_1 \\ y_1, & \eta_3, & \eta_1 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1 \\ \xi_2, & x_2, & x_3 \\ \eta_2, & y_2, & y_3 \end{vmatrix}$$

Or if the two sets of three points be called ABC, DEF,

$$ABC \times DEF = ADE \times FBC + AEF \times DBC + AFD \times BCE$$

is a relation between triangles.

The product of the two determinants

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & x_4 \\ y_1, & y_2, & y_3, & y_4 \\ z_1, & z_2, & z_3, & z_4 \end{vmatrix} \qquad \begin{vmatrix} 1, & 1, & 1, & 1 \\ \xi_1, & \xi_2, & \xi_3, & \xi_4 \\ \eta_1, & \eta_2, & \eta_3, & \eta_4 \\ \zeta_1, & \zeta_2, & \zeta_3, & \zeta_4 \end{vmatrix}$$

can be represented either as a sum of four terms

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & x_3, & \xi_1 \\ y_1, & y_2, & y_3, & \eta_1 \\ z_1, & z_2, & z_3, & \zeta_1 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1, & 1 \\ \xi_2, & \xi_3, & \xi_4, & x_4 \\ \eta_2, & \eta_3, & \eta_4, & y_4 \\ \zeta_2, & \zeta_3, & \zeta_4, & z_4 \end{vmatrix} + \dots,$$

or as the sum of six terms

$$\begin{vmatrix} 1, & 1, & 1, & 1 \\ x_1, & x_2, & \xi_1, & \xi_2 \\ y_1, & y_2, & \eta_1, & \eta_2 \\ z_1, & z_2, & \zeta_1, & \zeta_2 \end{vmatrix} \begin{vmatrix} 1, & 1, & 1, & 1 \\ \xi_3, & \xi_4, & x_3, & x_4 \\ \eta_3, & \eta_4, & y_3, & y_4 \\ \zeta_3, & \zeta_4, & z_3, & z_4 \end{vmatrix} + \dots.$$

Or calling the two sets of points $ABCD$, $EFGH$, we have the identical relations between the volumes of tetrahedra:

$$\begin{aligned} ABCD \times EFGH = {} & ABCE \times FGHD - ABCF \times GHED \\ & + ABCG \times HEFD - ABCH \times FGED \\ ABCD \times EFGH = {} & ABEF \times GHCD + ABGH \times EFCD \\ & + ABEG \times HFCD + ABHF \times EGCD \\ & + ABEH \times FGCD + ABFG \times EHCD. \end{aligned}$$

Application of Alternate Numbers in Geometry.

7. In applying alternate numbers to geometry, a number stands for a point in a flat space whose dimensions are one less than the number of units.

To begin with a plane, the units e_1, e_2, e_3 stand for the vertices of a fundamental triangle ABC. Any other number

$$P = xe_1 + ye_2 + ze_3$$

stands for some point in the plane of the triangle. It is generally convenient to assume that

$$x + y + z = 1,$$

so that x, y, z may be taken to mean the ratios of the triangles PBC, PCA, PAB to the triangle ABC, though this is not necessary.

If P and Q are two points, then

$$\frac{mP + nQ}{m + n}$$

is a point in the line PQ, dividing PQ in the ratio $n : m$. Thus $\frac{1}{2}(P + Q)$ is the middle point, and $P - Q$ the point at infinity of PQ.

Similar definitions hold for a space of three dimensions. Four points $ABCD$ being taken and represented by the units e_1, e_2, e_3, e_4 any other point in the space is represented by

$$P = xe_1 + ye_2 + ze_3 + we_4,$$

where if we choose we may write

$$x + y + z + w = 1,$$

x being the ratio of the tetrahedron $PBCD$ to $ABCD$.

And so on for a space of any number of dimensions.

Then a binary product $e_r e_s$ is a unit length measured on the line joining the points e_r, e_s or the distance between the points e_r, e_s.

A ternary product $e_r e_s e_t$ is a unit area measured on the plane of the points e_r, e_s, e_t, or the area of the triangle formed by the points e_r, e_s, e_t. And so on.

In a space of two dimensions the product of three points is the area of the triangle they form referred to the fundamental triangle.

Now if

$$P = x_1e_1 + y_1e_2 + z_1e_3,$$
$$Q = x_2e_1 + \dots$$
$$R = x_3e_1 + \dots$$
$$PQR = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} e_1e_2e_3.$$

And $e_1e_2e_3 = ABC = \Delta$, the area of the fundamental triangle, so that in areal co-ordinates

$$PQR = \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \Delta.$$

Similarly in a flat space of three dimensions if

$$e_1e_2e_3e_4 = V$$

is the volume of the fundamental tetrahedron, the volume of the tetrahedron formed by four points is

$$PQRS = \begin{vmatrix} x_1, & y_1, & z_1, & w_1 \\ x_2, & y_2, & z_2, & w_2 \\ x_3, & y_3, & z_3, & w_3 \\ x_4, & y_4, & z_4, & w_4 \end{vmatrix} V.$$

Similar definitions may be stated with reference to flat spaces of more than three dimensions.

The assumption which has been made throughout the present work, that the product of all the units of a system is unity, receives here its justification and explanation. For, geometrically speaking, the product of the units is the measure of the fundamental figure of the space considered, which is our unit of measure. In a plane, for example, it is the area of the triangle of reference, in ordinary space of three dimensions the volume of the tetrahedron of reference. It is no part of the plan of the present treatise to develop the geometrical applications of alternate numbers; for these we must refer to the memoirs and works of Grassmann and Schlegel.

Angles between straight lines. Solid angles. Spherical figures.

8. With rectangular axes let

$$\begin{array}{lcl} l_1, \ m_1, \ n_1 & \qquad & \lambda_1, \ \mu_1, \ \nu_1 \\ l_2, \ m_2, \ n_2 & & \lambda_2, \ \mu_2, \ \nu_2 \\ \cdots\cdots & & \cdots\cdots \end{array}$$

be the direction cosines of two sets of straight lines, then

$$\cos(ik) = l_i\lambda_k + m_i\mu_k + n_i\nu_k$$

is the cosine of the angle between the ith line of the first and kth of the second system; and, by compounding the two arrays, we get the determinant

$$|\cos(ik)|.$$

Hence by v. 3, if there are two sets of four straight lines we get

$$\begin{vmatrix} \cos(11) \ldots \cos(14) \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \cos(41) \ldots \cos(44) \end{vmatrix} = 0 \ldots\ldots\ldots\ldots\ldots \text{(i)}.$$

If there are two sets of three straight lines $a, b, c; f, g, h$,

$$\begin{vmatrix} \cos af, & \cos ag, & \cos ah \\ \cos bf, & \cos bg, & \cos bh \\ \cos cf, & \cos cg, & \cos ch \end{vmatrix} = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \begin{vmatrix} \lambda_1, & \mu_1, & \nu_1 \\ \lambda_2, & \mu_2, & \nu_2 \\ \lambda_3, & \mu_3, & \nu_3 \end{vmatrix}$$

$$= \sin(abc)\sin(fgh) \ldots\ldots\ldots\ldots \text{(ii)}.$$

If there are only two straight lines in each set

$$\begin{vmatrix} \cos(11), & \cos(12) \\ \cos(21), & \cos(22) \end{vmatrix} = \begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} \begin{vmatrix} \lambda_1, & \mu_1 \\ \lambda_2, & \mu_2 \end{vmatrix} + \ldots.$$

Now if n, ν be the directions of the shortest distances between the lines of each pair, and θ, ϕ the angles between the pairs,

$$\begin{vmatrix} l_1, & m_1 \\ l_2, & m_2 \end{vmatrix} = \sin\theta\cos(nz), \text{ \&c.}$$

$$\therefore \begin{vmatrix} \cos(11), & \cos(12) \\ \cos(21), & \cos(22) \end{vmatrix} = \sin\theta\sin\phi\cos(n\nu) \ldots\ldots \text{(iii)}.$$

9. If in the relation (i) of Art. 8 the two sets of straight lines coincide with one set of straight lines a, b, c, d, we have

$$\begin{vmatrix} 1 & , \cos(ab), & \cos(ac), & \cos(ad) \\ \cos(ba), & 1 \quad , & \cos(bc), & \cos(bd) \\ \cos(ca), & \cos(cb), & 1 \quad , & \cos(cd) \\ \cos(da), & \cos(db), & \cos(dc) & 1 \end{vmatrix} = 0.$$

This is the identical relation between the mutual inclination of four straight lines in space, or also the relation between the sides and diagonals of a spherical quadrilateral.

If we write $-\cos(AB)$ for $\cos(ab)$, or what comes to the same thing change the signs of the elements in the leading diagonal, it becomes the identical relation between the cosines of the dihedral angles of a tetrahedron formed by four planes A, B, C, D perpendicular to the lines a, b, c, d.

10. If the two straight lines marked 1 coincide with two straight lines u, v; while those marked 2, 3, 4 coincide with a set of oblique axes x, y, z,

$$\begin{vmatrix} \cos uv, & \cos ux, & \cos uy, & \cos uz \\ \cos xv, & 1\ \ , & \cos xy, & \cos xz \\ \cos yv, & \cos yx, & 1\ \ , & \cos yz \\ \cos zv, & \cos zx, & \cos zy, & 1 \end{vmatrix} = 0,$$

which gives the cosine of the angle between two straight lines u, v, referred to a set of oblique axes x, y, z, in terms of their direction cosines.

11. As another example of the use of the same formula, let ABC, $A'B'C'$ be two spherical triangles, O, O' the centres of the small circles circumscribing them. For our two sets of straight lines take the lines joining the centre to $O'ABC$, $OA'B'C'$. Then if $OO' = \phi$, and R, R' are the radii of the circumscribing circles, we get

$$\begin{vmatrix} \cos\phi, & \cos R', & \cos R', & \cos R' \\ \cos R, & \cos(AA'), & \cos(AB'), & \cos(AC') \\ \cos R, & \cos(BA'), & \cos(BB'), & \cos(BC') \\ \cos R, & \cos(CA'), & \cos(CB'), & \cos(CC') \end{vmatrix} = 0.$$

We can write this

$$\cos\phi \sin(ABC)\sin(A'B'C') = -\cos R\cos R' \begin{vmatrix} 0, & 1 & \dots & 1 \\ 1, & \cos(AA') & \dots & \cos(AC') \\ \dots & \dots & \dots & \dots \\ 1, & \cos(CA') & \dots & \cos(CC') \end{vmatrix}$$

If the angle at which the small circles cut is ψ

$$\cos\phi = \cos R \cos R' - \sin R \sin R' \cos\psi\,;$$

and the above formula can be written

$$(1 - \tan R \tan R' \cos\psi) \sin(ABC) \sin(A'B'C')$$

$$= - \begin{vmatrix} 0, & 1 & \ldots & 1 \\ 1, & \cos(AA') & \ldots & \cos(AC') \\ \cdots & \cdots & \cdots & \cdots \\ 1, & \cos(CA') & \ldots & \cos(CC') \end{vmatrix}.$$

If the two systems coincide $\psi = \pi$, and we get

$$\begin{vmatrix} \sec^2 R, & 1, & 1, & 1 \\ 1, & 1, & \cos c, & \cos b \\ 1, & \cos c, & 1, & \cos a \\ 1, & \cos b, & \cos a, & 1 \end{vmatrix} = 0,$$

a, b, c being the sides of the spherical triangle.

12. Similar relations can be developed in the same way for a plane.

In a plane we can shew that for two sets of three straight lines

$$\begin{vmatrix} \cos(11), & \cos(12), & \cos(13) \\ \cos(21), & \cos(22), & \cos(23) \\ \cos(31), & \cos(32), & \cos(33) \end{vmatrix} = 0,$$

and then deduce

$$\begin{vmatrix} 1, & \cos C, & \cos B \\ \cos C, & 1, & \cos A \\ \cos B, & \cos A, & 1 \end{vmatrix} = 0, \qquad \begin{vmatrix} \cos(xy), & \cos(xa), & \cos(xb) \\ \cos(ay), & 1, & \cos(ab) \\ \cos(by), & \cos(ba), & 1 \end{vmatrix} = 0,$$

similar to the equations in Arts. 9 and 10.

13. Next, let us compound two arrays

$$\begin{matrix} 1, \ l_1, \ m_1, \ n_1 \\ \cdots\cdots\cdots \\ 1, \ l_p, \ m_p, \ n_p \end{matrix} \qquad \begin{matrix} 1, -\lambda_1, -\mu_1, -\nu_1 \\ \cdots\cdots\cdots \\ 1, -\lambda_p, -\mu_p, -\nu_p. \end{matrix}$$

We get the determinant

$$|\, 1 - \cos(ik) \,| = |\, 2 \sin^2 \tfrac{1}{2}(ik) \,|\,.$$

Hence, by v. 3, for two sets of five straight lines

$$\begin{vmatrix} \sin^2\tfrac{1}{2}(11) \dots \sin^2\tfrac{1}{2}(15) \\ \dots\dots\dots\dots\dots\dots \\ \sin^2\tfrac{1}{2}(51) \dots \sin^2\tfrac{1}{2}(55) \end{vmatrix} = 0 \quad \dots\dots\dots\dots\dots\dots\text{(i)}.$$

For two sets of four straight lines a, b, c, d; a', b', c', d',

$$16 \begin{vmatrix} \sin^2\tfrac{1}{2}(aa')\dots\sin^2\tfrac{1}{2}(ad') \\ \dots\dots\dots\dots\dots\dots \\ \sin^2\tfrac{1}{2}(da')\dots\sin^2\tfrac{1}{2}(dd') \end{vmatrix} = -\,|\,1, l_i, m_i, n_i\,| \times |\,1, \lambda_i, \mu_i, \nu_i\,| \quad (i=1, 2, 3, 4) \dots\dots\dots\text{(ii)}.$$

Expanding the determinants on the right according to the elements of their first column, our determinant

$$= \{\sin(bcd) + \sin(cad) + \sin(abd) - \sin(abc)\}$$
$$\times \{\sin(b'c'd') + \sin(c'a'd') + \sin(a'b'd') - \sin(a'b'c')\}.$$

For two sets of three straight lines, our determinant is

$$\begin{vmatrix} 1-\cos(11) \dots 1-\cos(13) \\ \dots\dots\dots\dots\dots\dots \\ 1-\cos(31) \dots 1-\cos(33) \end{vmatrix},$$

or

$$\begin{vmatrix} 1, & 0 & \dots & 0 \\ 1, & 1-\cos(11) & \dots & 1-\cos(13) \\ 1, & \dots & \dots & \dots \\ 1, & 1-\cos(31) & \dots & 1-\cos(33) \end{vmatrix} = \begin{vmatrix} 1, & -1, & \dots & -1 \\ 1, & -\cos(11) & \dots & -\cos(13) \\ 1, & \dots & \dots & \dots \\ 1, & -\cos(31) & \dots & -\cos(33) \end{vmatrix}.$$

This is equal to the sum of the products of determinants of the third order taken from the two arrays. Omitting the term

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \begin{vmatrix} -\lambda_1, & -\mu_1, & -\nu_1 \\ -\lambda_2, & -\mu_2, & -\nu_2 \\ -\lambda_3, & -\mu_3, & -\nu_3 \end{vmatrix} = \begin{vmatrix} -\cos(11) \dots -\cos(13) \\ \dots\dots\dots\dots\dots\dots \\ -\cos(31) \dots -\cos(33) \end{vmatrix},$$

we get

$$\begin{vmatrix} 0, & 1 & \dots & 1 \\ 1, & \cos(11) & \dots & \cos(13) \\ \dots & \dots & \dots & \dots \\ 1, & \cos(31) & \dots & \cos(33) \end{vmatrix} = |\,1, l, m\,|\,|\,1, \lambda, \mu\,| + |\,1, l, n\,|\,|\,1, \lambda, \nu\,| + |\,1, m, n\,|\,|\,1, \mu, \nu\,|.$$

If the straight lines be called a, b, c; a', b', c', and N_1, N_2, N_3

are the directions of the shortest distances between bc, ca, ab, we have

$$|1, l, m| = \sin(bc)\cos(N_1z) + \sin(ca)\cos(N_2z) + \sin(ab)\cos(N_3z),$$

$$|1, \lambda, \mu| = \sin(b'c')\cos(N_1'z) + \sin(c'a')\cos(N_2'z) + \sin(a'b')\cos(N_3'z),$$

and similarly for the other determinants. In particular, if $a\,b\,c$ lie in one plane, and $a'b'c'$ in another, the normals to the two planes being N, N', the value of the determinant is

$$\{\sin(bc)+\sin(ca)+\sin(ab)\}\{\sin(b'c')+\sin(c'a')+\sin(a'b')\}\cos(NN'),$$

viz. this

$$= -\begin{vmatrix} 0, & 1 & \dots & 1 \\ 1, & \cos(aa') & \dots & \cos(ac') \\ \dots & \dots & \dots & \dots \\ 1, & \cos(ca') & \dots & \cos(cc') \end{vmatrix} \quad \dots\dots\dots\dots\dots\dots\text{(iii)}.$$

For two sets of two straight lines we deduce in the same way, if R, r are the directions of the external bisectors between them,

$$\begin{vmatrix} 0, & 1, & 1 \\ 1, & \cos(11), & \cos(12) \\ 1, & \cos(21), & \cos(22) \end{vmatrix} = -4\sin\frac{ab}{2}\sin\frac{a'b'}{2}\,.\cos(Rr).$$

14. If we compound the arrays

$$\begin{matrix} l_1, & m_1, & n_1, & 1, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ l_i, & m_i, & n_i, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{matrix} \qquad \begin{matrix} \lambda_1, & \mu_1, & \nu_1, & 0, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \lambda_i, & \mu_i, & \nu_i, & 0, & 1 \\ 0, & 0, & 0, & 1, & 0, \end{matrix}$$

we get the determinant

$$\begin{vmatrix} \cos(11) & \dots & \cos(1i), & 1 \\ \dots & \dots & \dots & \dots \\ \cos(i1) & \dots & \cos(ii), & 1 \\ 1 & \dots & 1, & 0 \end{vmatrix}.$$

Hence for two sets of five straight lines

$$\begin{vmatrix} \cos(11) & \dots & \cos(15), & 1 \\ \dots & \dots & \dots & \dots \\ \cos(51) & \dots & \cos(55), & 1 \\ 1 & \dots & 1 & \end{vmatrix} = 0.$$

For two sets of four lines

$$\begin{vmatrix} \cos(11) \ldots \cos(14), & 1 \\ \cdots\cdots\cdots\cdots & \\ \cos(41) \ldots \cos(44), & 1 \\ 1 \quad \ldots \quad 1, & 0 \end{vmatrix} = -|1, l, m, n|\,|1, \lambda, \mu, \nu|,$$

and so on.

But these are not new theorems. In the first, for example, if we expand by IV. 24, according to products of elements in the last row and column, each term vanishes by Art. 8.

On Systems of Straight Lines.

15. If
$$\frac{x-p}{\cos\alpha} = \frac{y-q}{\cos\beta} = \frac{z-r}{\cos\gamma}$$
be the equations of a straight line, then

$$a = \cos\alpha, \qquad b = \cos\beta, \qquad c = \cos\gamma,$$

$$f = \begin{vmatrix} q & r \\ \cos\beta, & \cos\gamma \end{vmatrix}, \quad g = \begin{vmatrix} r & p \\ \cos\gamma, & \cos\alpha \end{vmatrix}, \quad h = \begin{vmatrix} p & q \\ \cos\alpha, & \cos\beta \end{vmatrix}$$

are called the co-ordinates of the line. It is plain that

$$af + bg + ch = 0.$$

16. If the constants belonging to two straight lines be denoted by the suffixes 1 and 2, the equation of a plane through the second line, parallel to the first, is

$$\begin{vmatrix} x-p_2, & y-q_2, & z-r_2 \\ \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \end{vmatrix} = 0.$$

If d be the shortest distance between the two straight lines, and θ the angle between them, it follows that

$$d\sin\theta = \begin{vmatrix} p_1-p_2, & q_1-q_2, & r_1-r_2 \\ \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \end{vmatrix}$$

$$= \begin{vmatrix} p_1, & q_1, & r_1 \\ \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \end{vmatrix} + \begin{vmatrix} p_2, & q_2, & r_2 \\ \cos\alpha_2, & \cos\beta_2, & \cos\gamma_2 \\ \cos\alpha_1, & \cos\beta_1, & \cos\gamma_1 \end{vmatrix}$$

$$= a_2f_1 + b_2g_1 + c_2h_1 + a_1f_2 + b_1g_2 + c_1h_2.$$

If the expression on the right vanishes, then either $d = 0$, i.e. the two straight lines intersect, or $\sin\theta = 0$, when they are parallel, and hence also meet. It is convenient to have a name for the expression on the right. If a unit force acted in one of the lines its moment about the other would be $d \sin\theta$, i.e. in terms of the co-ordinates of the lines

$$a_1 f_2 + b_1 g_2 + c_1 h_2 + a_2 f_1 + b_2 g_1 + c_2 h_1 .$$

Hence we shall call this the moment of the two straight lines. If two straight lines meet their moment vanishes.

17. Let us take two systems of straight lines whose co-ordinates are

$$\begin{array}{ll} a_1,\, b_1,\, c_1,\, f_1,\, g_1,\, h_1 & \quad f_1',\, g_1',\, h_1',\, a_1',\, b_1',\, c_1' \\ \cdots\cdots\cdots\cdots & \quad \cdots\cdots\cdots\cdots \\ a_i,\, b_i,\, c_i,\, f_i,\, g_i,\, h_i & \quad f_i',\, g_i',\, h_i',\, a_i',\, b_i',\, c_i' . \end{array}$$

Then if m_{rs} denotes the moment of the line r of the first and s of the second system, by compounding the two arrays we get the determinant

$$| \; m_{ik} \; | .$$

Hence for two sets of seven straight lines

$$\begin{vmatrix} m_{11} \ldots m_{17} \\ \cdots\cdots\cdots \\ m_{71} \ldots m_{77} \end{vmatrix} = 0,$$

an identical relation between the mutual moments of two sets of seven straight lines. If the two systems coincide

$$\begin{vmatrix} 0, & m_{12} & \ldots & m_{17} \\ m_{21}, & 0 & \ldots & m_{27} \\ \cdots & \cdots & \cdots & \cdots \\ m_{71}, & m_{72} & \ldots & 0 \end{vmatrix} = 0.$$

For two sets of six straight lines

$$\begin{vmatrix} m_{11} \ldots m_{16} \\ \cdots\cdots \;\; \cdots\cdots \\ m_{61} \ldots m_{66} \end{vmatrix} = \begin{array}{l} | \; a_i,\, b_i,\, c_i,\, f_i,\, g_i,\, h_i \; | \\ \times \, | \; f_i',\, g_i',\, h_i',\, a_i',\, b_i',\, c_i' \; | \end{array} \qquad (i = 1, 2 \ldots 6).$$

If one of the sets of six straight lines—say the first—is met by a common transversal whose co-ordinates are a, b, c, f, g, h, we have for each of the straight lines of that system

$$af_i + bg_i + ch_i + fa_i + gb_i + hc_i = 0.$$

Thus the first of the determinants on the right vanishes, and

$$\begin{vmatrix} m_{11} \dots m_{16} \\ \dots\dots\dots \\ m_{61} \dots m_{66} \end{vmatrix} = 0$$

is the relation between the mutual moments of two sets of six straight lines, one set of which is met by a common transversal.

If the two sets coincide we get the identity for a system of six lines met by a common transversal.

18. If the moments of a system of forces about one set of seven lines be $m_1, m_2 \dots m_7$, and about a second set $n_1, n_2 \dots n_7$, we can establish an identity among the moments involved.

For if any force P of the system act in a line whose co-ordinates are a, b, c, f, g, h, we have

$$\begin{aligned} m_1 &= \Sigma P\ \{af_1 + bg_1 + ch_1 + fa_1 + gb_1 + hc_1\} \\ &= f_1\Sigma Pa + g_1\Sigma Pb + h_1\Sigma Pc + a_1\Sigma Pf + b_1\Sigma Pg + c_1\Sigma Ph, \end{aligned}$$

and six other equations for $m_2 \dots m_7$. Hence eliminating

$$\Sigma Pa,\ \Sigma Pb \dots \Sigma Ph,$$

we get

$$\begin{vmatrix} m_1, a_1, b_1, c_1, f_1, g_1, h_1 \\ \dots\dots\dots\dots\dots\dots \\ m_7, a_7, b_7, c_7, f_7, g_7, h_7 \end{vmatrix} = 0,$$

and a similar equation for the other system. Hence each of the determinants

$$\begin{vmatrix} 0, m_1, a_1, b_1, c_1, f_1, g_1, h_1 \\ \dots\dots\dots\dots\dots\dots\dots \\ 0, m_7, a_7, b_7, c_7, f_7, g_7, h_7 \\ 1, 0, 0, 0, 0, 0, 0, 0 \end{vmatrix} \quad \begin{vmatrix} n_1, 0, f_1', g_1', h_1', a_1', b_1', c_1' \\ \dots\dots\dots\dots\dots\dots\dots \\ n_7, 0, f_7', g_7', h_7', a_7', b_7', c_7' \\ 0, 1, 0, 0, 0, 0, 0, 0 \end{vmatrix}$$

vanishes. Forming their product we get

$$\begin{vmatrix} m_{11} \dots m_{17}, n_1 \\ \dots\dots\dots\dots \\ m_{71} \dots m_{77}, n_7 \\ m_1 \dots m_7, 0 \end{vmatrix} = 0.$$

Tetrahedra and Triangles.

19. Let there be two systems of points in space whose coordinates referred to rectangular axes are (x_i, y_i, z_i), (ξ_i, η_i, ζ_i). Let us compound the two arrays

$$\begin{array}{lcl} x_1, y_1, z_1, 1, 0 & & -2\xi_1, -2\eta_1, -2\zeta_1, 0, 1 \\ \dots\dots\dots\dots & & \dots\dots\dots\dots\dots\dots\dots \\ x_i, y_i, z_i, 1, 0 & & -2\xi_i, -2\eta_i, -2\zeta_i, 0, 1 \\ 0,\ 0,\ 0,\ 0,\ 1 & & 0, \quad 0, \quad 0, \quad 1, 0, \end{array}$$

we obtain the determinant

$$\begin{vmatrix} c_{11} & \dots & c_{1i}, & 1 \\ \dots & \dots & \dots & \dots \\ c_{i1} & \dots & c_{ii}, & 1 \\ 1 & \dots & 1 & \end{vmatrix},$$

where $$c_{rs} = -2x_r\xi_s - 2y_r\eta_s - 2z_r\zeta_s.$$

To the rth row add the last multiplied by $x_r^2 + y_r^2 + z_r^2$, and to the sth column add the last multiplied by $\xi_s^2 + \eta_s^2 + \zeta_s^2$, the determinant is unaltered and its elements are now

$$\begin{aligned} d_{rs} &= x_r^2 + y_r^2 + z_r^2 - 2x_r\xi_s - 2y_r\eta_s - 2z_r\zeta_s + \xi_s^2 + \eta_s^2 + \zeta_s^2 \\ &= (x_r - \xi_s)^2 + (y_r - \eta_s)^2 + (z_r - \zeta_s)^2, \end{aligned}$$

i.e. d_{rs} is the square of the distance between the rth point of the first and sth point of the second system. We have then the determinant

$$\begin{vmatrix} d_{11} & \dots & d_{1i}, & 1 \\ \dots & \dots & \dots & \dots \\ d_{i1} & \dots & d_{ii}, & 1 \\ 1 & \dots & 1 & \end{vmatrix}.$$

If $i = 5$ the determinant vanishes, hence

$$\begin{vmatrix} d_{11} & \dots & d_{15}, & 1 \\ \dots & \dots & \dots & \dots \\ d_{51} & \dots & d_{55}, & 1 \\ 1 & \dots & 1 & \end{vmatrix} = 0 \quad \dots\dots\dots\dots\dots\dots \text{(i)}$$

is the identical relation which subsists between the lines joining two sets of five points in space. If the two systems coincide $d_{ii} = 0$, and the determinant, which is then symmetrical, gives the relation between the lines joining five points in space. The relation in this form is due to Cayley.

If $i=4$,

$$\begin{vmatrix} d_{11}\ldots d_{14}, & 1 \\ \cdots\cdots\cdots & \\ d_{41}\ldots d_{44}, & 1 \\ 1 \;\ldots\; 1, & 0 \end{vmatrix} = \begin{vmatrix} x_1, & y_1, & z_1, & 1, & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ x_4, & y_4, & z_4, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1 \end{vmatrix} \times \begin{vmatrix} -2\xi_1, & -2\eta_1, & -2\zeta_1, & 0, & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ -2\xi_4, & -2\eta_4, & -2\zeta_4, & 0, & 1 \\ 0, & 0, & 0, & 1, & 0 \end{vmatrix}$$

$$= 288\,VV' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

where V, V' are the volumes of the tetrahedra formed by the two sets of four points.

If the two sets coincide in a single tetrahedron, for which a, a'; b, b'; c, c' are pairs of opposite edges,

$$288\,V^2 = \begin{vmatrix} 0, & c'^2, & b'^2, & a'^2, & 1 \\ c'^2, & 0, & c^2, & b^2, & 1 \\ b'^2, & c^2, & 0, & a^2, & 1 \\ a'^2, & b^2, & a^2, & 0, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}.$$

If $i=3$, we have

$$\begin{vmatrix} d_{11}\ldots d_{13}, & 1 \\ \cdots\cdots\cdots & \\ d_{31}\ldots d_{33}, & 1 \\ 1 \;\ldots\; 1, & 0 \end{vmatrix} = -4|x,y,1|\,|\xi,\eta,1| - 4|x,z,1|\,|\xi,\zeta,1| - 4|y,z,1|\,|\eta,\zeta,1|,$$

all the other determinants on the right vanishing identically.

Now if Δ, Δ' be the areas of the triangles formed by the two sets of three points, (l, m, n), (λ, μ, ν) the direction cosines of the normals to their planes,

$$|x, y, 1| = \text{twice projection of } \Delta \text{ on plane } xy = 2\Delta n,$$

and similarly for the others; hence if ϕ is the angle between the planes of the triangles

$$\begin{vmatrix} d_{11}\ldots d_{13}, & 1 \\ \cdots\cdots\cdots & \\ d_{31}\ldots d_{33}, & 1 \\ 1 \;\ldots\; 1, & 0 \end{vmatrix} = -16\Delta\Delta'\cos\phi \ldots\ldots\ldots\ldots\text{(iii)}.$$

Lastly, if $i=2$,

$$\begin{vmatrix} d_{11}, & d_{12}, & 1 \\ d_{21}, & d_{22}, & 1 \\ 1, & 1, & 0 \end{vmatrix} = \begin{vmatrix} x_1, & 1, & 0 \\ x_2, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix} \begin{vmatrix} -2\xi_1, & 0, & 1 \\ -2\xi_2, & 0, & 1 \\ 0, & 1, & 0 \end{vmatrix} + \ldots$$

$$= 2(x_1-x_2)(\xi_1-\xi_2) + 2(y_1-y_2)(\eta_1-\eta_2) + 2(z_1-z_2)(\zeta_1-\zeta_2),$$

the other terms vanishing. Now if a, b be the lengths of the lines joining the points of the first and second systems and θ the angle between them,

$$\frac{x_1 - x_2}{a} \cdot \frac{\xi_1 - \xi_2}{b} + \dots + \dots = \cos\theta.$$

Hence

$$\begin{vmatrix} d_{11}, & d_{12}, & 1 \\ d_{21}, & d_{22}, & 1 \\ 1, & 1, & 0 \end{vmatrix} = 2ab\cos\theta \dots\dots\dots\dots\dots\dots\text{(iv)}.$$

20. If in case (iii) of Art. 19 we allow the two sets of three points to coincide with the vertices of a single triangle whose sides are a, b, c,

$$-16\Delta^2 = \begin{vmatrix} 0, & c^2, & b^2, & 1 \\ c^2, & 0, & a^2, & 1 \\ b^2, & a^2, & 0, & 1 \\ 1, & 1, & 1, & 0 \end{vmatrix}.$$

Multiply each column by abc, then

$$-16\Delta^2 a^4b^4c^4 = \begin{vmatrix} 0, & abc^3, & ab^3c, & abc \\ abc^3, & 0, & a^3bc, & abc \\ ab^3c, & a^3bc, & 0, & abc \\ abc, & abc, & abc, & 0 \end{vmatrix}.$$

Divide the first, second, and third rows and columns by bc, ca, ab respectively, then

$$-16\Delta^2 = \begin{vmatrix} 0, & c, & b, & a \\ c, & 0, & a, & b \\ b, & a, & 0, & c \\ a, & b, & c, & 0 \end{vmatrix}$$

$$= \begin{vmatrix} a, & b, & c, & 0 \\ b, & a, & 0, & c \\ c, & 0, & a, & b \\ 0, & c, & b, & a \end{vmatrix}$$

by an interchange of columns.

If in the first expression for $-16\Delta^2$ we divide the second and

third columns by a^2, and then multiply the first and last rows by a^2, we get:

$$-16\Delta^2 = \begin{vmatrix} 0, & c^2, & b^2, & a^2 \\ c^2, & 0, & 1, & 1 \\ b^2, & 1, & 0, & 1 \\ a^2, & 1, & 1, & 0 \end{vmatrix}.$$

21. If in case (ii) of Art. 19 one of the sets of four points—say the first—lies in a plane, then $V = 0$, and

$$\begin{vmatrix} d_{11} \dots d_{14}, & 1 \\ \dots\dots\dots\dots & \\ d_{41} \dots d_{44}, & 1 \\ 1 \quad \dots \quad 1 & \end{vmatrix} = 0.$$

If one of the sets in case (iii) lies in a straight line the corresponding triangle vanishes; hence

$$\begin{vmatrix} d_{11} \dots d_{13}, & 1 \\ \dots\dots\dots\dots & \\ d_{31} \dots d_{33}, & 1 \\ 1 \quad \dots \quad 1 & \end{vmatrix} = 0.$$

By allowing the second system to coincide with the first we get the identical relations between the lines joining four coplanar and three collinear points.

22. In the identical relation

$$\begin{vmatrix} d_{11} \dots d_{15}, & 1 \\ \dots\dots\dots\dots & \\ d_{51} \dots d_{55}, & 1 \\ 1 \quad \dots \quad 1 & \end{vmatrix} = 0$$

between the squares of the lines joining two sets of five points, let the fifth point of the first system be the centre of the sphere circumscribing the tetrahedron formed by the first four points of the second system, and the point 5 of the second system the centre of the sphere circumscribing the first four points of the first system. Then

$$d_{15} = d_{25} = d_{35} = d_{45} = R^2$$
$$d_{51} = d_{52} = d_{53} = d_{54} = R'^2.$$

Also, if ϕ be the angle at which the two circumscribing spheres intersect,

$$d_{55} = R^2 + R'^2 + 2RR' \cos \phi.$$

Hence with an interchange of rows and columns

$$\begin{vmatrix} d_{11} \dots d_{14}, & 1, & R^2 \\ \dots\dots\dots\dots & & \\ d_{41} \dots d_{44}, & 1, & R^2 \\ 1 \;\dots\; 1, & 0, & 1 \\ R'^2 \dots R'^2, & 1, & d_{55} \end{vmatrix} = 0.$$

Multiply the fifth column by R^2 and subtract it from the last, and the fifth row by R'^2 and subtract it from the last, then

$$\begin{vmatrix} d_{11} \dots d_{14}, & 1, & 0 \\ \dots\dots\dots\dots & & \\ d_{41} \dots d_{44}, & 1, & 0 \\ 1 \;\dots\; 1, & 0, & 1 \\ 0 \;\dots\; 0, & 1, & 2RR' \cos\phi \end{vmatrix} = 0.$$

Or, resolving according to the elements of the last row and column, we have by Art. 19 (ii)

$$576 VRV'R' \cos \phi = \begin{vmatrix} d_{11} \dots d_{14} \\ \dots\dots\dots \\ d_{41} \dots d_{44} \end{vmatrix}.$$

We see from this that so long as the circumscribing spheres remain fixed the tetrahedra can turn about in them without altering the value of the determinant on the right. The determinant vanishes if the circumscribing spheres of the two systems cut orthogonally. This relation is due to Siebeck.

23. If in Art. 22 we allow the two tetrahedra to coincide we get, since $\phi = \pi$,

$$16 (6VR)^2 = - \begin{vmatrix} 0, & a'^2, & b'^2, & c'^2 \\ a'^2, & 0, & c^2, & b^2 \\ b'^2, & c^2, & 0, & a^2 \\ c'^2, & b^2, & a^2, & 0 \end{vmatrix}.$$

Multiply the second, third and fourth rows and columns by a^2, b^2, c^2 respectively, then

$$16\,(6VR)^2\,a^4b^4c^4 = -\begin{vmatrix} 0, & (aa')^2, & (bb')^2, & (cc')^2 \\ (aa')^2, & 0, & a^2b^2c^2, & a^2b^2c^2 \\ (bb')^2, & a^2b^2c^2, & 0, & a^2b^2c^2 \\ (cc')^2, & a^2b^2c^2, & a^2b^2c^2, & 0 \end{vmatrix}.$$

Divide the second, third and fourth rows by $(abc)^2$, then multiply the first column by the same quantity,

$$16\,(6VR)^2 = -\begin{vmatrix} 0, & (aa')^2, & (bb')^2, & (cc')^2 \\ (aa')^2, & 0, & 1, & 1 \\ (bb')^2, & 1, & 0, & 1 \\ (cc')^2, & 1, & 1, & 0 \end{vmatrix}.$$

Now if we write

$$aa' = kx, \quad bb' = ky, \quad cc' = kz,$$

then if Δ is the area of the triangle, whose sides are x, y, z, we have by Art. 20,

$$(6VR)^2 = k^4\Delta^2,$$
$$6VR = k^2\Delta.$$

This triangle, whose sides are proportional to the square roots of the products of pairs of opposite sides of the tetrahedron, has many interesting relations to the tetrahedron. It is sometimes called the conjugate triangle.

Formulæ relating to the Ellipsoid.

24. Let (x_i, y_i, z_i) and (ξ_i, η_i, ζ_i) be two sets of points on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Then, if d_{rs} denote the square of the distance between the rth and sth points of the two systems and D_{rs} the square of the parallel semidiameter, we have

$$a_{rs} = \frac{d_{rs}}{D_{rs}} = 2\left(1 - \frac{x_r\xi_s}{a^2} - \frac{y_r\eta_s}{b^2} - \frac{z_r\zeta_s}{c^2}\right).$$

Hence, if we compound the two arrays,

$$\begin{array}{lll} \frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c},\ 1 & \qquad & -\frac{2\xi_1}{a},\ -\frac{2\eta_1}{b},\ -\frac{2\zeta_1}{c},\ 2 \\ \dots\dots\dots\dots & & \dots\dots\dots\dots\dots\dots \\ \frac{x_i}{a},\ \frac{y_i}{b},\ \frac{z_i}{c},\ 1 & & -\frac{2\xi_i}{a},\ -\frac{2\eta_i}{b},\ -\frac{2\zeta_i}{c},\ 2, \end{array}$$

we get as in the preceding articles:—

For two sets of five points situated on the ellipsoid,

$$\begin{vmatrix} a_{11} \dots a_{15} \\ \dots\dots\dots \\ a_{51} \dots a_{55} \end{vmatrix} = 0.$$

For two sets of four points forming two tetrahedra of volumes V, V',

$$\begin{vmatrix} a_{11} \dots a_{14} \\ \dots\dots\dots \\ a_{41} \dots a_{44} \end{vmatrix} = -\frac{576VV'}{a^2b^2c^2}.$$

Similar formulæ can be established for an ellipse in a plane.

If the ellipsoid become a sphere, $a=b=c=R$, and since all diameters are equal, we can replace a_{rs} by d_{rs}. Thus

$$\begin{vmatrix} d_{11} \dots d_{15} \\ \dots\dots\dots \\ d_{51} \dots d_{55} \end{vmatrix} = 0$$

is an identical relation between two sets of five points on a sphere. This relation is due to Cayley.

The second relation in this case reduces to the result of Art. 22, when the two tetrahedra have the same circumscribing sphere.

25. If the points $(x_i, y_i, z_i)(\xi_i, \eta_i, \zeta_i)$ are not situated on the ellipsoid, then since

$$a_{rs} = \frac{d_{rs}}{D_{rs}} = \frac{(x_r-\xi_s)^2}{a^2} + \frac{(y_r-\eta_s)^2}{b^2} + \frac{(z_r-\zeta_s)^2}{c^2}$$

$$= \frac{x_r^2}{a^2} + \frac{y_r^2}{b^2} + \frac{z_r^2}{c^2} - \frac{2x_r\xi_s}{a^2} - \frac{2y_r\eta_s}{b^2} - \frac{2z_r\zeta_s}{c^2} + \frac{\xi_s^2}{a^2} + \frac{\eta_s^2}{b^2} + \frac{\zeta_s^2}{c^2};$$

if we compound the two arrays whose rth rows are

$$\frac{x_i^2}{a^2}+\frac{y_i^2}{b^2}+\frac{z_i^2}{c^2},\ \frac{x_i}{a},\ \frac{y_i}{b},\ \frac{z_i}{c},\ 1,$$

$$1,\ \frac{-2\xi_i}{a},\ \frac{-2\eta_i}{b},\ \frac{-2\zeta_i}{c},\ \frac{\xi_i^2}{a^2}+\frac{\eta_i^2}{b^2}+\frac{\zeta_i^2}{c^2},$$

we get the identical relation (v. 3)

$$\begin{vmatrix} a_{11} \dots a_{16} \\ \dots\dots\dots \\ a_{61} \dots a_{66} \end{vmatrix} = 0$$

for any two systems of six points in space, and

$$\begin{vmatrix} a_{11} \dots a_{15} \\ \dots\dots\dots \\ a_{51} \dots a_{55} \end{vmatrix} = \left|\frac{x_i^2}{a^2}+\frac{y_i^2}{b^2}+\frac{z_i^2}{c^2},\ \frac{x_i}{a},\ \frac{y_i}{b},\ \frac{z_i}{c},\ 1\right| \times \left|1,\ \frac{-2\xi_i}{a},\ \frac{-2\eta_i}{b},\ \frac{-2\zeta_i}{c},\ \frac{\xi_i^2}{a^2}+\frac{\eta_i^2}{b^2}+\frac{\zeta_i^2}{c^2}\right|$$

$$(i=1,\ 2 \dots 5),$$

for any two systems of five points.

If in the latter equation all the points of the first system lie on the ellipsoid

$$\left(\frac{x-p}{a}\right)^2+\left(\frac{y-q}{b}\right)^2+\left(\frac{z-r}{c}\right)^2=m^2,$$

we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-\frac{2px}{a^2}-\frac{2qy}{b^2}-\frac{2rz}{c^2}+\frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}=m^2$$

satisfied for each point of the system. Hence we see by eliminating

$$\frac{-2p}{a},\ \frac{-2q}{b},\ \frac{-2r}{c},\ \frac{p^2}{a^2}+\frac{q^2}{b^2}+\frac{r^2}{c^2}-m^2$$

between these five equations, that the first determinant on the right vanishes. Hence

$$\begin{vmatrix} a_{11} \dots a_{15} \\ \dots\dots\dots \\ a_{51} \dots a_{55} \end{vmatrix} = 0,$$

if the five points of one of the systems lie on an ellipsoid similar

and similarly situated to the given one. If the ellipsoid reduce to a sphere, we get

$$\begin{vmatrix} d_{11} \dots d_{16} \\ \dots\dots\dots \\ d_{61} \dots d_{66} \end{vmatrix} = 0,$$

an identical and homogeneous relation between the lines joining two sets of six points. And

$$\begin{vmatrix} d_{11} \dots d_{15} \\ \dots\dots\dots \\ d_{51} \dots d_{55} \end{vmatrix} = 0$$

for five points situated on a sphere.

26. In like manner, if for the same systems of points as in the last article we compound the arrays

$$\begin{array}{lllll} \frac{x_1}{a}, & \frac{y_1}{b}, & \frac{z_1}{c}, & 1, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \frac{x_i}{a}, & \frac{y_i}{b}, & \frac{z_i}{c}, & 1, & 0 \\ 0, & 0, & 0, & 0, & 1, \end{array} \qquad \begin{array}{lllll} -\frac{2\xi_1}{a}, & -\frac{2\eta_1}{b}, & -\frac{2\zeta_1}{c}, & 0, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ -\frac{2\xi_i}{a}, & -\frac{2\eta_i}{b}, & -\frac{2\zeta_i}{c}, & 0, & 1 \\ 0, & 0, & 0, & 1, & 0 \end{array}$$

we get the determinant

$$\begin{vmatrix} c_{11} \dots c_{1i}, & 1 \\ \dots\dots\dots & \\ c_{i1} \dots c_{ii}, & 1 \\ 1 \;\dots\; 1 & \end{vmatrix}$$

where $$c_{rs} = -\frac{2x_r\xi_s}{a^2} - \frac{2y_r\eta_s}{b^2} - \frac{2z_r\zeta_s}{c^2}.$$

Multiply the last column by

$$\frac{\xi_s^2}{a^2} + \frac{\eta_s^2}{b^2} + \frac{\zeta_s^2}{c^2}$$

and add it to the sth column, and the last row by

$$\frac{x_r^2}{a^2} + \frac{y_r^2}{b^2} + \frac{z_r^2}{c^2}$$

and add it to the rth row, then the element at the intersection of the rth row and sth column is

$$\left(\frac{x_r - \xi_s}{a}\right)^2 + \left(\frac{y_r - \eta_s}{b}\right)^2 + \left(\frac{z_r - \zeta_s}{c}\right)^2 = a_{rs}.$$

And hence (v. 3),

$$\begin{vmatrix} a_{11} \ldots a_{15}, & 1 \\ \ldots\ldots\ldots\ldots & \\ a_{51} \ldots a_{55}, & 1 \\ 1 \ldots 1 & \end{vmatrix} = 0$$

is an identical relation between any two sets of five points in space. If the ellipsoid becomes a sphere we regain Cayley's relation (Art. 19, i).

For $i = 4$, we have

$$\begin{vmatrix} a_{11} \ldots a_{14}, & 1 \\ \ldots\ldots\ldots\ldots & \\ a_{41} \ldots a_{44}, & 1 \\ 1 \ldots 1 & \end{vmatrix} = \frac{288\,VV'}{a^2b^2c^2},$$

V, V' being the volumes of the tetrahedra formed by each set of four points.

27. The polar plane of a point $P(x_r, y_r, z_r)$ with respect to the ellipsoid is

$$\frac{xx_r}{a^2} + \frac{yy_r}{b^2} + \frac{zz_r}{c^2} = 1.$$

The distance of a point $Q(\xi_s, \eta_s, \zeta_s)$ from this plane is

$$(P, Q) = -p\left(\frac{x_r\xi_s}{a^2} + \frac{y_r\eta_s}{b^2} + \frac{z_r\zeta_s}{c^2} - 1\right)$$

$$\frac{1}{p^2} = \frac{x_r^2}{a^4} + \frac{y_r^2}{b^4} + \frac{z_r^2}{c^4}.$$

If (Q, P) and q denote like quantities for the point Q,

$$\frac{(P, Q)}{p} = \frac{(Q, P)}{q} = 1 - \frac{x_r\xi_s}{a^2} - \frac{y_r\eta_s}{b^2} - \frac{z_r\zeta_s}{c^2}.$$

This function has been called by Faure the index of the two

points P and Q; let it be denoted by I_{rs}. Then, by compounding the arrays whose ith rows are

$$\frac{x_i}{a},\ \frac{y_i}{b},\ \frac{z_i}{c},\ 1; \qquad \frac{-\xi_i}{a},\ \frac{-\eta_i}{b},\ \frac{-\zeta_i}{c},\ 1;$$

we obtain

$$\begin{vmatrix} I_{11} \dots I_{15} \\ \dots\dots\dots \\ I_{51} \dots I_{55} \end{vmatrix} = 0$$

$$\begin{vmatrix} I_{11} \dots I_{14} \\ \dots\dots\dots \\ I_{41} \dots I_{44} \end{vmatrix} = -\frac{36VV'}{a^2b^2c^2}.$$

28. It may be remarked that these space relations connected with an ellipsoid are not really more general than those connected with a sphere. For they may be deduced from the latter by applying to the whole configuration the homogeneous pure strain which changes the sphere

$$x^2 + y^2 + z^2 = R^2$$

to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Formulæ relating to Systems of Spheres.

29. If r, s be the radii of two spheres, ϕ the angle at which they intersect, and d the distance between their centres, then

$$d^2 = r^2 + s^2 + 2rs \cos \phi.$$

The function

$$2rs \cos \phi = d^2 - r^2 - s^2$$

is of importance in the study of the mutual relations of spheres; it is called the power of the two spheres. We shall denote it by p_{rs}.

If one of the spheres, say s, becomes a point, the limit of $2rs \cos \phi$ is $d^2 - r^2$, i.e. the square of the tangent from the point to the sphere, or what is known as the power of the sphere at the point, or the power of the point with respect to the sphere.

If both spheres reduce to points the limit of $2rs\cos\phi$ is d^2, the square of the distance between the points.

If one of the spheres becomes a plane, and p is its distance from the centre of the other,

$$\cos\phi = \frac{p}{r}.$$

If the second sphere become a point, and p is its distance from the plane, the limit of $r\cos\phi$ is p.

30. Let (x_i, y_i, z_i) and (ξ_k, η_k, ζ_k) be the co-ordinates of the centres of two spheres of radii r_i and ρ_k, then if p_{ik} is their mutual power

$$\begin{aligned} p_{ik} &= d^2 - r_i^2 - \rho_k^2 \\ &= x_i^2 + y_i^2 + z_i^2 - r_i^2 - 2x_i\xi_k - 2y_i\eta_k - 2z_i\zeta_k + \xi_k^2 + \eta_k^2 + \zeta_k^2 - \rho_k^2. \end{aligned}$$

Hence, compounding the two arrays

$$\begin{matrix} x_1,\ y_1,\ z_1,\ 1,\ x_1^2+y_1^2+z_1^2-r_1^2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_i,\ y_i,\ z_i,\ 1,\ x_i^2+y_i^2+z_i^2-r_i^2, \end{matrix}$$

and

$$\begin{matrix} -2\xi_1,\ -2\eta_1,\ -2\zeta_1,\ \xi_1^2+\eta_1^2+\zeta_1^2-\rho_1^2,\ 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -2\xi_i,\ -2\eta_i,\ -2\zeta_i,\ \xi_i^2+\eta_i^2+\zeta_i^2-\rho_i^2,\ 1, \end{matrix}$$

we see by v. 3 that for two systems of six spheres

$$\begin{vmatrix} p_{11} \cdots p_{16} \\ \cdots\cdots \\ p_{61} \cdots p_{66} \end{vmatrix} = 0 \ \cdots\cdots\cdots\cdots\cdots\cdots\text{(i)}.$$

If $\cos\phi_{ik}$ is the cosine of the angle at which two spheres cut, we can also write this

$$|\cos\phi_{ik}| = 0 \qquad (i, k = 1, 2 \ldots 6).$$

For two systems, each of five spheres,

$$\begin{vmatrix} p_{11} \cdots p_{15} \\ \cdots\cdots \\ p_{51} \cdots p_{55} \end{vmatrix} \ \cdots\cdots\cdots\cdots\cdots\cdots\text{(ii)}$$

$$= |x, y, z, 1, x^2+y^2+z^2-r^2| \times |-2\xi, -2\eta, -2\zeta, \xi^2+\eta^2+\zeta^2-\rho^2, 1|.$$

If the five spheres of one of the systems—say the first—have a common radical centre, taking this for origin we should have

$$x^2 + y^2 + z^2 - r^2 = c^2,$$

where c is the same for all the five spheres. Hence, in the first determinant on the right of (ii), the fourth and fifth columns are proportionals and the determinant vanishes.

Thus

$$\begin{vmatrix} p_{11} \dots p_{15} \\ \dots\dots\dots \\ p_{51} \dots p_{55} \end{vmatrix} = 0 \qquad \text{(iii)}$$

when the five spheres of one system have a common radical centre.

If the five spheres of the first system reduce to points (iii) is the condition that they should lie on a sphere.

If both systems reduce to points we regain Cayley's condition, that the five points of one system should lie on the same sphere.

31. But if neither of the determinants on the right of (ii) vanish, expand the first determinant with regard to the elements of the last column.

Then $$p_i = x_i^2 + y_i^2 + z_i^2 - r_i^2$$

is the power of the origin (i.e. any point) with regard to the ith sphere of the first system. Then if we write 1, 2, 3, 4, 5 for the centres of the five spheres, and denote by

$$v_1 = (2345), \quad v_2 = (3451), \text{ \&c.},$$

the volumes of the tetrahedra formed by the points in brackets, and if accents denote similar quantities for the second determinant, we have in place of (ii)

$$|p_{ik}| = 288\,(v_1 p_1 + v_2 p_2 + \dots + v_5 p_5)\,(v_1' p_1' + \dots + v_5' p_5') \qquad (i, k = 1, 2 \dots 5).$$

Now describe about the origin a sphere of radius r, cutting the spheres $r_1 \dots r_5$ at angles $\phi_1 \dots \phi_5$.

We have, since (Art. 5)

$$v_1 + v_2 + \dots + v_5 = 0 \text{ identically,}$$

$$\begin{aligned} v_1 p_1 + \dots + v_5 p_5 &= v_1 (p_1 - r^2) + \dots + v_5 (p_5 - r^2) \\ &= 2r\,(v_1 r_1 \cos\phi_1 + \dots + v_5 r_5 \cos\phi_5), \end{aligned}$$

and ρ being a similar sphere for the second system,

$$|p_{ik}| = 288\rho r \Sigma 2 v_i r_i \cos\phi_i \Sigma 2 v_i' \rho_i \cos\phi_i' \qquad (i, k = 1 \dots 5).$$

Thus $r\Sigma 2v_i r_i \cos\phi_i$ is independent of the particular sphere r; let this be the orthotomic sphere of the first four, and let R denote its radius; then this sum reduces to

$$2v_5 r_5 R \cos(r_5 R),$$

and the second factor, in like manner, becomes

$$2v_5' \rho_5 R' \cos(\rho_5 R').$$

Hence

$$\begin{vmatrix} p_{11} \cdots p_{15} \\ \cdots\cdots\cdots \\ p_{51} \cdots p_{55} \end{vmatrix} = 1152 v_5 v_5' r_5 \rho_5 R R' \cos(r_5 R)\cos(\rho_5 R').$$

32. For the fifth sphere of each system in this last equation take the orthotomic sphere of the first four spheres in the other system. Then in the determinant on the left all the elements in the last row and column vanish except p_{55}, and

$$p_{55} = 2RR'\cos(RR').$$

Hence we obtain

$$\begin{vmatrix} p_{11} \cdots p_{14} \\ \cdots\cdots\cdots \\ p_{41} \cdots p_{44} \end{vmatrix} 2RR'\cos(RR') = 1152 v_5 v_5' R^2 R'^2 \cos^2(RR'),$$

or dividing out the common factors and writing V, V' for v_5, v_5', we get for two sets of four spheres

$$\begin{vmatrix} p_{11} \cdots p_{14} \\ \cdots\cdots\cdots \\ p_{41} \cdots p_{44} \end{vmatrix} = 576\, VV'RR'\cos(RR').$$

If the spheres reduce to points we regain Siebeck's formula (Art. 22).

The determinant on the left vanishes if the orthotomic spheres of the two systems of spheres cut orthogonally.

33. To determine the meaning of the determinant

$$|p_{ik}| \qquad (i, k = 1, 2, 3).$$

In the determinant of Art. 32, let the fourth sphere of each system be the plane determined by the centres of the first three spheres

of the other system, then if Δ, Δ' be the areas of the triangles formed by the centres, ϕ the angle between their planes,

$$\lim. \frac{V'}{r_4'} = 3\Delta \cos \phi, \quad \lim. \frac{V}{r_4} = 3\Delta' \cos \phi.$$

Also if the radical axis of the spheres of the first system meet the plane of centres of the second system in P, whose power with reference to the spheres is p, and P', p' denote like quantities for the other system,

$$2RR' \cos (RR') = PP'^2 - p - p'.$$

Hence

$$\begin{vmatrix} p_{11} \dots p_{13} \\ \dots\dots\dots \\ p_{31} \dots p_{33} \end{vmatrix} = 16\Delta\Delta' \cos \phi \, (PP'^2 - p - p').$$

34. In the relations

$$\begin{vmatrix} d_{11} \dots d_{15}, & 1 \\ \dots\dots\dots & \\ d_{51} \dots d_{55}, & 1 \\ 1 \dots 1 & \end{vmatrix} = 0,$$

$$\begin{vmatrix} d_{11} \dots d_{14}, & 1 \\ \dots\dots\dots & \\ d_{41} \dots d_{44}, & 1 \\ 1 \dots 1 & \end{vmatrix} = -288VV',$$

of Art. 19, let us suppose the sets of points to be the centres of our spheres.

Then if we multiply the last column by ρ_i^2 and subtract it from the ith column, and the last row by r_k^2 and subtract it from the kth row, we get the relations

$$\begin{vmatrix} p_{11} \dots p_{15}, & 1 \\ \dots\dots\dots & \\ p_{51} \dots p_{55}, & 1 \\ 1 \dots 1 & \end{vmatrix} = 0,$$

$$\begin{vmatrix} p_{11} \dots p_{14}, & 1 \\ \dots\dots\dots & \\ p_{41} \dots p_{44}, & 1 \\ 1 \dots 1 & \end{vmatrix} = -288VV',$$

connecting the mutual powers of two sets of five spheres and two sets of four spheres.

35. Another element connected with two spheres is the length of their common tangent. For two spheres of radii r, s the distance between whose centres is d and which cut at an angle ϕ, the square of the length of the common tangent is given by

$$\begin{aligned} t &= d^2 - (r-s)^2 \\ &= 2rs\cos^2 \tfrac{1}{2}\phi. \end{aligned}$$

If one sphere reduce to a point, t is the power of that point with respect to the other sphere. If both spheres reduce to points, t is the square of the distance between them.

36. Using the same notation as in Art. 30, if t_{ik} is the square of the tangent common to the two spheres

$$\begin{aligned} t_{ik} &= (x_i-\xi_k)^2 + (y_i-\eta_k)^2 + (z_i-\zeta_k)^2 - (r_i-\rho_k)^2 \\ &= x_i^2+y_i^2+z_i^2-r_i^2-2x_i\xi_k-2y_i\eta_k-2z_i\zeta_k+2r_i\rho_k+\xi_k^2+\eta_k^2+\zeta_k^2-\rho_k^2. \end{aligned}$$

Hence, compounding the two arrays

$$\begin{array}{c} x_1,\ y_1,\ z_1,\ r_1,\ 1,\ x_1^2+y_1^2+z_1^2-r_1^2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ x_i,\ y_i,\ z_i,\ r_i,\ 1,\ x_i^2+y_i^2+z_i^2-r_i^2 \\ 0,\ 0,\ 0,\ 0,\ 0,\ \quad 1 \end{array}$$

$$\begin{array}{c} -2\xi_1,\ -2\eta_1,\ -2\zeta_1,\ 2\rho_1,\ \xi_1^2+\eta_1^2+\zeta_1^2-\rho_1^2,\ 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ -2\xi_i,\ -2\eta_i,\ -2\zeta_i,\ 2\rho_i,\ \xi_i^2+\eta_i^2+\zeta_i^2-\rho_i^2,\ 1 \\ 0,\quad 0,\quad 0,\quad 0,\quad 1,\quad 0, \end{array}$$

we get for two systems of six spheres the identity

$$\begin{vmatrix} t_{11} \dots t_{16}, & 1 \\ \cdots\cdots & \\ t_{61} \dots t_{66}, & 1 \\ 1 \quad \dots & 1 \end{vmatrix} = 0.$$

For two systems of five spheres we should get

$$\begin{vmatrix} t_{11} \dots t_{15}, & 1 \\ \cdots\cdots & \\ t_{51} \dots t_{55}, & 1 \\ 1 \quad \dots & 1 \end{vmatrix} = 576\,(v_1r_1 + \dots + v_5r_5)\,(v_1'\rho_1 + \dots + v_5'\rho_5),$$

using the notation of Art. 31.

If t_5 is the angle at which the plane of similitude of the first four spheres of the first system cuts each of these spheres, and (r_5t_5) the angle at which it cuts the fifth sphere, and similarly for the second system, we can reduce this to the form

$$\begin{vmatrix} t_{11} \dots t_{15}, 1 \\ \dots\dots\dots\dots \\ t_{51} \dots t_{55}, 1 \\ 1 \dots 1 \end{vmatrix} = 576 v_5 r_5 v_5' \rho_5 \left(1 - \frac{\cos(r_5t_5)}{\cos t_5}\right)\left(1 - \frac{\cos(\rho_5\tau_5)}{\cos \tau_5}\right).$$

Hence the determinant vanishes if one of the systems of five spheres has a common plane of similitude.

For two sets of four spheres, after some reduction we can prove that

$$\begin{vmatrix} t_{11} \dots t_{14}, 1 \\ \dots\dots\dots\dots \\ t_{41} \dots t_{44}, 1 \\ 1 \dots 1 \end{vmatrix} = 288 vv' \left(1 - \frac{\cos\phi}{\cos t \cos \tau}\right),$$

where ϕ is the angle between the planes of similitude of the two systems, and t, τ the angles at which they cut their sets of spheres.

37. By compounding the arrays whose ith rows are

$$x_i,\ y_i,\ z_i,\ r_i,\ 1,\ x_i^2 + y_i^2 + z_i^2 - r_i^2$$

and $$-2\xi_i,\ -2\eta_i,\ -2\zeta_i,\ 2\rho_i,\ \xi_i^2 + \eta_i^2 + \zeta_i^2 - \rho_i^2,\ 1,$$

we get the homogeneous relation between the sets of tangents common to two sets of seven spheres

$$\begin{vmatrix} t_{11} \dots t_{17} \\ \dots\dots\dots \\ t_{71} \dots t_{77} \end{vmatrix} = 0.$$

38. We may make use of this last relation to solve the problem: Determine the equation of the sphere having with five given spheres tangents of the same length.

Let the equations of the five given spheres be

$$S_1 = 0 \dots\dots S_5 = 0.$$

Take these for the first five of each set of spheres in Art. 38, let the sixth sphere be the one required, and the seventh a point on the sixth.

Then we shall have

$$t_{67}=0, \quad t_{7i}=S_i, \quad t_{6i}=k,$$

and the equation is

$$\begin{vmatrix} 0, & t_{12}, & t_{13}, & t_{14}, & t_{15}, & 1, & S_1 \\ t_{21}, & 0, & t_{23}, & t_{24}, & t_{25}, & 1, & S_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ t_{51}, & t_{52}, & t_{53}, & t_{54}, & 0, & 1, & S_5 \\ 1, & 1, & 1, & 1, & 1, & 0, & 0 \\ S_1, & S_2, & S_3, & S_4, & S_5, & 0, & 0 \end{vmatrix} = 0.$$

This is apparently of the fourth order, but by means of the sixth row and column we can get rid of the terms of the second degree in the seventh row and column.

39. All the equations of this section relating to spheres are capable of numerous and varied applications, some of these will be found in the examples, and others in the memoirs of Bauer, Darboux and Frobenius.

EXAMPLES.

PROVE the following relations: 1—5.

1.
$$\begin{vmatrix} (b+c)^2, & ab\ , & ac \\ ab\ , & (c+a)^2, & bc \\ ac\ , & bc\ , & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3,$$

$$\begin{vmatrix} (b+c)^2, & c^2\ , & b^2 \\ c^2\ , & (c+a)^2, & a^2 \\ b^2\ , & a^2\ , & (a+b)^2 \end{vmatrix} = 2\,(bc+ca+ab)^3.$$

2.
$$\begin{vmatrix} 1\ , & 1\ , & 1 \\ \tan A\ , & \tan B\ , & \tan C \\ \sin 2A, & \sin 2B\ , & \sin 2C \end{vmatrix} = 0,$$

if A, B, C are the angles of a triangle.

3.
$$\begin{vmatrix} 1, & x, & (a+x)\sqrt{(c+x)} \\ 1, & y, & (a+y)\sqrt{(c+y)} \\ 1, & z, & (a+z)\sqrt{(c+z)} \end{vmatrix} = 0,$$

if $$\tan^{-1}\sqrt{\left(\frac{a-c}{c+x}\right)} + \tan^{-1}\sqrt{\left(\frac{a-c}{c+y}\right)} + \tan^{-1}\sqrt{\left(\frac{a-c}{c+z}\right)} = 0.$$

4.
$$\begin{vmatrix} 1\ , & \cos\alpha\ , & \cos(\alpha+\beta), & \cos(\alpha+\beta+\gamma), & \cos(\alpha+\beta+\gamma+\delta) \\ \cos\alpha\ , & 1\ , & \cos\beta\ , & \cos(\beta+\gamma)\ , & \cos(\beta+\gamma+\delta) \\ \cos(\alpha+\beta)\ , & \cos\beta\ , & 1\ , & \cos\gamma\ , & \cos(\gamma+\delta) \\ \cos(\alpha+\beta+\gamma), & \cos(\beta+\gamma)\ , & \cos\gamma\ , & 1\ , & \cos\delta \\ \cos(\alpha+\beta+\gamma+\delta), & \cos(\beta+\gamma+\delta), & \cos(\gamma+\delta), & \cos\delta\ , & 1 \end{vmatrix} = 0.$$

5.
$$\begin{vmatrix} a+b+c+d, & a-b-c+d, & a-b+c-d \\ a-b-c+d, & a+b+c+d, & a+b-c-d \\ a-b+c-d, & a+b-c-d, & a+b+c+d \end{vmatrix}$$
$$= 16\,(bcd+acd+abd+abc).$$

6. If a, b, c are the sides of a triangle of area Δ, $2s = a + b + c$, then

$$\begin{vmatrix} (b+c)^2, & ab, & ac, & a \\ ab, & (c+a)^2, & bc, & b \\ ac, & bc, & (a+b)^2, & c \\ a, & b, & c & \end{vmatrix} = -16s\Delta(a^2r_1 + b^2r_2 + c^2r_3),$$

r_1, r_2, r_3 being the radii of the escribed circles.

If the elements in the principal diagonal are $(b-c)^2$, &c., the other elements being as before, the value of the determinant is

$$-16\frac{\Delta^3}{s}\left(\frac{a^2}{r_1} + \frac{b^2}{r_2} + \frac{c^2}{r_3}\right).$$

$$\begin{vmatrix} (b+c)^2, & ab, & ac, & a \\ ab, & (c+a)^2, & bc, & b \\ ac, & bc, & (a+b)^2, & c \\ 1, & 1, & 1 & \end{vmatrix} = -16s\Delta(ar_1 + br_2 + cr_3),$$

$$\begin{vmatrix} (b+c)^2, & ab, & ac, & 1 \\ ab, & (c+a)^2, & bc, & 1 \\ ac, & bc, & (a+b)^2, & 1 \\ 1, & 1, & 1 & \end{vmatrix} = 16\Delta^2 - 20abcs.$$

7. If $S = a_1 + a_2 + \ldots + a_n$, $A_i = S - a_i$, prove the following theorems:

$$\begin{vmatrix} x - A_1, & a_2 & \ldots & a_n \\ a_1, & x - A_2 & \ldots & a_n \\ \ldots & \ldots & \ldots & \ldots \\ a_1, & a_2 & \ldots & x - A_n \end{vmatrix} = x(x-S)^{n-1},$$

$$\begin{vmatrix} x - a_1, & A_2 & \ldots & A_n \\ A_1, & x - a_2 & \ldots & A_n \\ \ldots & \ldots & \ldots & \ldots \\ A_1, & A_2 & \ldots & x - a_n \end{vmatrix} = \{x + (n-2)S\}(x-S)^{n-1}.$$

8. The determinant

$$\begin{vmatrix} a, & b, & b, & b \ldots\ldots \\ a, & b, & a, & a \ldots\ldots \\ b, & b, & a, & b \ldots\ldots \\ a, & a, & a, & b \ldots\ldots \\ \ldots & \ldots & \ldots & \ldots\ldots \end{vmatrix}$$

(the diagonal consisting of a and b alternately and each row being filled up with the other letter) is equal to

$$(-1)^{n-1}(n-1)(a-b)^{2n}.$$

The determinant is supposed to have $2n$ rows.

9. If in a determinant all the minors of the second order are divisible by the same quantity p, then the minors of the mth order are divisible by p^{m-1}.

10. If in a determinant of the nth order there be a block of p by q elements all of which are divisible by a, the determinant is divisible by a^{p+q-n}.

11. Prove the theorems:

$$\begin{vmatrix} a, & b, & c, & d, & \dots \\ a, & a+b, & a+b+c, & a+b+c+d, & \dots \\ a, & 2a+b, & 3a+2b+c, & 4a+3b+2c+d, & \dots \\ a, & 3a+b, & 6a+3b+c, & 10a+6b+3c+d, & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = a^n,$$

$$\begin{vmatrix} a, & b, & c, & d \dots \\ a, & a+b, & a+2b+c, & a+3b+3c+d \dots \\ a, & 2a+b, & 4a+4b+c, & 8a+12b+6c+d \dots \\ a, & 3a+b, & 9a+6b+c, & 27a+27b+9c+d \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} = a^n\, 1^{n-1}.\, 2^{n-2}.\, 3^{n-3} \dots (n-1),$$

where $a, b, c, d \dots$ are any quantities whatever, and n is the order of the determinant. In the first determinant each row after the first is obtained from the preceding by the rule that the rth element of any row is the sum of the first r elements of the preceding row. In the second determinant the rth element of any row is the sum of the first r elements of the preceding row multiplied respectively by the coefficients in the expansion of $(1+x)^{r-1}$.

12. If $$D = \begin{vmatrix} a, & b, & c, & d \dots \\ -a, & b, & p, & q \dots \\ -a, & -b, & c, & r \dots \\ -a, & -b, & -c, & d \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} \ (n \text{ rows}),$$

then $$D = 2^{n-1}\, abcd \dots$$

The elements of the first row and leading diagonal are $a, b, c, d \dots$; in each column the elements below the leading diagonal are equal to the element in the first row but of opposite sign, the others are any whatever.

13. If $$D = \begin{vmatrix} \cos n\alpha_0, & \cos(n-1)\,\alpha_0 \dots \cos\alpha_0, & 1 \\ \cos n\alpha_1, & \cos(n-1)\,\alpha_1 \dots \cos\alpha_1, & 1 \\ \dots & \dots & \dots \\ \cos n\alpha_n, & \cos(n-1)\,\alpha_n \dots \cos\alpha_n, & 1 \end{vmatrix},$$

$$D_1 = \begin{vmatrix} \cos^n \alpha_0, & \cos^{n-1}\alpha_0 \ldots \cos \alpha_0, & 1 \\ \cos^n \alpha_1, & \cos^{n-1}\alpha_1 \ldots \cos \alpha_1, & 1 \\ \ldots\ldots & \ldots\ldots & \ldots \\ \cos^n \alpha_n, & \cos^{n-1}\alpha_n \ldots \cos \alpha_n, & 1 \end{vmatrix},$$

$$D_2 = \begin{vmatrix} \sin(n+1)\alpha_0, & \sin n\alpha_0 \ldots \sin \alpha_0 \\ \sin(n+1)\alpha_1, & \sin n\alpha_1 \ldots \sin \alpha_1 \\ \ldots\ldots & \ldots\ldots \\ \sin(n+1)\alpha_n, & \sin n\alpha_n \ldots \sin \alpha_n \end{vmatrix},$$

then

$$\frac{D}{D_1} = 2^{\frac{n(n-1)}{2}}, \qquad \frac{D_2}{D_1} = 2^{\frac{n(n+1)}{2}} \sin \alpha_0 \sin \alpha_1 \ldots \alpha_n.$$

14. If $b_{ik} = (a_{i1} + a_{i2} + \ldots + a_{in}) - a_{ik}$, then

$$\begin{vmatrix} b_{11} \ldots b_{1n} \\ \ldots\ldots \\ b_{n1} \ldots b_{nn} \end{vmatrix} = (-1)^{n-1}(n-1) \begin{vmatrix} a_{11} \ldots a_{1n} \\ \ldots\ldots \\ a_{n1} \ldots a_{nn} \end{vmatrix}.$$

But if $$b_{ik} = (a_{i1} + a_{i2} + \ldots + a_{in}) - 2a_{ik}$$

$$\begin{vmatrix} b_{11} \ldots b_{1n} \\ \ldots\ldots \\ b_{n1} \ldots b_{nn} \end{vmatrix} = (n-2)(-2)^{n-1} \begin{vmatrix} a_{11} \ldots a_{1n} \\ \ldots\ldots \\ a_{n1} \ldots a_{nn} \end{vmatrix}.$$

15. Prove that every power of a symmetrical determinant may be expressed as a symmetrical determinant of the same order.

16. If for each element a_{ik} of a determinant A we write in turn $a_{ik} + c$, we get n^2 new determinants. If these be taken as the elements of another determinant its value will be

$$(Ac)^{n-1}(c + S),$$

where S is the sum of all the elements of A.

17. If $$u = (X_1 - a_1 b_1)(X_2 - a_2 b_2) \ldots (X_n - a_n b_n),$$
prove that the value of the determinant

$$\begin{vmatrix} X_1, & a_2 b_1, & a_3 b_1 & \ldots & a_n b_1 \\ a_1 b_2, & X_2, & a_3 b_2 & \ldots & a_n b_2 \\ a_1 b_3, & a_2 b_3, & X_3 & \ldots & a_n b_3 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1 b_n, & a_2 b_n, & a_3 b_n & \ldots & X_n \end{vmatrix}$$

is $$u \left\{ 1 + \frac{a_1 b_1}{X_1 - a_1 b_1} + \ldots + \frac{a_n b_n}{X_n - a_n b_n} \right\},$$

and the value of

$$\begin{vmatrix} 0, & a_1, & a_2 & \dots & a_n \\ b_1, & X_1, & a_2 b_1 & \dots & a_n b_1 \\ b_2, & a_1 b_2, & X_2 & \dots & a_n b_2 \\ \dots & \dots & \dots & \dots & \dots \\ b_n, & a_1 b_n, & a_2 b_n & \dots & X_n \end{vmatrix}$$

is
$$-u\left\{\frac{a_1 b_1}{X_1 - a_1 b_1} + \dots + \frac{a_n b_n}{X_n - a_n b_n}\right\}.$$

18. If $u = (x-2a_1)(x-2a_2) \dots (x-2a_n)$, prove the following theorems:

$$\begin{vmatrix} (x-a_1)^2, & a_2^2 & , & a_3^2 & \dots \\ a_1^2 & , (x-a_2)^2, & & a_3^2 & \dots \\ a_1^2 & , & a_2^2 & , (x-a_3)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = x^{n-1} u \left\{x + \Sigma \frac{a_i^2}{x-2a_i}\right\},$$

$$\begin{vmatrix} 0, & 1 & , & 1 & , & 1 & \dots \\ 1, & (x-a_1)^2, & & a_2^2 & , & a_3^2 & \dots \\ 1, & a_1^2 & , & (x-a_2)^2, & & a_3^2 & \dots \\ 1, & a_1^2 & , & a_2^2 & , & (x-a_3)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = -x^{n-1} \frac{du}{dx},$$

$$\begin{vmatrix} (x-a_1)^2, & a_1 a_2 & , & a_1 a_3 & \dots \\ a_1 a_2 & , (x-a_2)^2, & & a_2 a_3 & \dots \\ a_1 a_3 & , & a_2 a_3 & , (x-a_3)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{vmatrix} = x^{n-1} u \left\{x + \Sigma \frac{a_i^2}{x-2a_i}\right\},$$

$$\begin{vmatrix} 0\,, & a_1 & , & a_2 & , & a_3 & \dots \\ a_1, & (x-a_1)^2, & & a_1 a_2 & , & a_1 a_3 & \dots \\ a_2, & a_1 a_2 & , & (x-a_2)^2, & & a_2 a_3 & \dots \\ a_3, & a_1 a_3 & , & a_2 a_3 & , & (x-a_3)^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} = -x^{n-1} u \Sigma \frac{a_i^2}{x-2a_i}.$$

And if

$$D = \begin{vmatrix} (x-a_1)^2, & a_2^2 & \dots & a_n^2 & , b_1, & 1 \\ a_1^2 & , (x-a_2)^2 & \dots & a_n^2 & , b_2, & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_1^2 & , \quad a_2^2 & \dots & (x-a_n)^2, & b_n, & 1 \\ b_1 & , \quad b_2 & \dots & b_n & , & \\ 1 & , \quad 1 & \dots & 1 & & \end{vmatrix},$$

then

$$\frac{D}{x^{n-2}u} = \left\{\frac{1}{x-2a_1} + \dots + \frac{1}{x-2a_n}\right\}\left\{\frac{b_1^2}{x-2a_1} + \dots + \frac{b_n^2}{x-2a_n}\right\} - \left\{\frac{b_1}{x-2a_1} + \dots + \frac{b_n}{x-2a_n}\right\}^2.$$

19. Prove that, if $S = x + y + z + u$,

$$\begin{vmatrix} (S-u)^2, & x^2, & y^2, & z^2 \\ u^2, & (S-x)^2, & y^2, & z^2 \\ u^2, & x^2, & (S-y)^2, & z^2 \\ u^2, & x^2, & y^2, & (S-z)^2 \end{vmatrix} = 2S^5xyzu\left\{\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{u} - \frac{4}{S}\right\}$$

$$\begin{vmatrix} 0, & 1, & 1, & 1, & 1 \\ 1, & (S-u)^2, & x^2, & y^2, & z^2 \\ 1, & u^2, & (S-x)^2, & y^2, & z^2 \\ 1, & u^2, & x^2, & (S-y)^2, & z^2 \\ 1, & u^2, & x^2, & y^2, & (S-z)^2 \end{vmatrix} = S^3\{x^2(y+z+u) + y^2(x+z+u) + z^2(x+y+u) + u^2(x+z+y) + 2xyz + 2xzu + 2yzu + 2xyu - x^3 - y^3 - z^3 - u^3\}.$$

20. If $X = \operatorname{cn} x \operatorname{dn} x$, &c. prove that

$$\begin{vmatrix} \operatorname{sn} x, & \operatorname{sn}^3 x, & X \\ \operatorname{sn} y, & \operatorname{sn}^3 y, & Y \\ \operatorname{sn} z, & \operatorname{sn}^3 z, & Z \end{vmatrix} = \operatorname{sn}(y-z)\operatorname{sn}(z-x)\operatorname{sn}(x-y)\operatorname{sn}(x+y+z)\,M$$

where

$$M = 1 - k^2\{\operatorname{sn}^2 y\operatorname{sn}^2 z + \operatorname{sn}^2 z\operatorname{sn}^2 x + \operatorname{sn}^2 x\operatorname{sn}^2 y\} + k^2(1+k^2)\operatorname{sn}^2 x\operatorname{sn}^2 y\operatorname{sn}^2 z - k^2\operatorname{sn} x\operatorname{sn} y\operatorname{sn} z\,(YZ\operatorname{sn} x + ZX\operatorname{sn} y + XY\operatorname{sn} z).$$

21. If $\operatorname{sn} x\operatorname{cn} x\operatorname{dn} x = X$, &c. prove that

$$\begin{vmatrix} 1, & \operatorname{sn}^2 x, & \operatorname{sn}^4 x, & X \\ 1, & \operatorname{sn}^2 y, & \operatorname{sn}^4 y, & Y \\ 1, & \operatorname{sn}^2 z, & \operatorname{sn}^4 z, & Z \\ 1, & \operatorname{sn}^2 u, & \operatorname{sn}^4 u, & U \end{vmatrix} = 0,$$

provided $x + y + z + u = 2pK + 2qiK'$,

p, q being integers.

22. If $S_{ij} = a_{ij} + a_{i+h,\,j+h} - a_{i+h,\,j} - a_{i,\,j+h}$,

then

$$\begin{vmatrix} S_{11}, & S_{12} & \dots & S_{1k-h} \\ S_{21}, & S_{22} & \dots & S_{2k-h} \\ \dots & \dots & \dots & \dots \\ S_{k-h1}, & S_{k-h2} & \dots & S_{k-hk-h} \end{vmatrix}$$

is the sum of all the minors of order $k-h$ of the determinant $A = |a_{ik}|$; excepting always in such sum those determinants and their complements of order h which in their formation have two row or column suffixes congruent with regard to the modulus h.

23. If

$$D_n = \begin{vmatrix} 0, & 1, & 1, & 1, & 1 & \dots \\ 1, & 0, & x, & 0, & 0 & \dots \\ 1, & y, & 0, & x, & 0 & \dots \\ 1, & 0, & y, & 0, & x & \dots \\ 1, & 0, & 0, & y, & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \ (n \text{ rows}),$$

where all elements are zeros, with the exception of the border, and two lines of elements one on each side of the principal diagonal, prove that

$$D_{2n} = -xy\, D_{2n-2} - \frac{x^{2n-1} + y^{2n-1}}{x+y},$$

$$D_{2n+1} = -xy\, D_{2n-1} + \frac{x^{2n} + y^{2n}}{x+y} - \frac{2(-xy)^n}{x+y},$$

and hence that

$$D_{2n} = -\left\{\frac{x^n - (-y)^n}{x+y}\right\}^2,$$

$$D_{2n+1} = \frac{x^{2n+1} + y^{2n+1} - (2n+1)(x+y)(-xy)^n}{(x+y)^2}.$$

24. If

$$D_n = \begin{vmatrix} c, & a, & c, & c, & c & \dots \\ b, & c, & a, & c, & c & \dots \\ c, & b, & c, & a, & c & \dots \\ c, & c, & b, & c, & a & \dots \\ c, & c, & c, & b, & c & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \ (n \text{ rows}),$$

where all the elements are c with the exception of two lines, one on either side of the principal diagonal, prove that

$$D_{2n-1} = c\left\{\frac{(a-c)^n - (c-b)^n}{a+b-2c}\right\}^2.$$

Find also the value of D_{2n}.

25. If

$$D_n = \begin{vmatrix} 0, & 1, & 1, & 1, & 1 & \dots \\ 1, & c, & a, & 0, & 0 & \dots \\ 1, & b, & c, & a, & 0 & \dots \\ 1, & 0, & b, & c, & a & \dots \\ 1, & 0, & 0, & b, & c & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{vmatrix} \ (n \text{ rows}),$$

(where, with the exception of the border, the elements in the leading diagonal are c, in the lines on either side of it a and b, the rest are zero), then

$$D_n - cD_{n-1} + abD_{n-2} = \frac{(-a)^{n-1} + (-b)^{n-1}}{a+b+c} - \frac{c}{a+b+c} \cdot \frac{u^{n-1} - v^{n-1}}{u-v} + \frac{2ab}{a+b+c} \cdot \frac{u^{n-2} - v^{n-2}}{u-v},$$

where u and v are the roots of the equation

$$z^2 - cz + ab = 0.$$

Hence shew that

$$D_n = \frac{u^n + v^n}{(a+b+c)^2} - \frac{nc\,(u^n + v^n)}{(a+b+c)\,(u-v)^2} + \frac{2abn}{a+b+c} \cdot \frac{u^{n-1} + v^{n-1}}{(u-v)^2} - \frac{(-a)^n + (-b)^n}{(a+b+c)^2}.$$

26. The value of the determinant

$$\begin{vmatrix} u_1, & u_2 & \dots & u_n \\ u_n, & u_1 & \dots & u_{n-1} \\ u_{n-1}, & u_n & \dots & u_{n-2} \\ \dots & \dots & \dots & \dots \\ u_2, & u_3 & \dots & u_1 \end{vmatrix}$$

(i) If $u_r = a + (r-1)\,b$ is

$$\frac{2a + (n-1)\,b}{2} (-nb)^{n-1}.$$

(ii) If $u_r = x^{r-1}$ is $(1-x^n)^{n-1}$.

(iii) If $u_r = r^2$ is

$$(-1)^{n-1} \frac{(n+1)\,(2n+1)\,n^{n-2}}{12} \{(n+2)^n - n^n\}.$$

(iv) If $u_r = \cos\{a + (r-1)b\}$ is

$$\frac{[\cos a - \cos(a+nb)]^n - [\cos(a-b) - \cos\{a+(n-1)b\}]^n}{2(1-\cos nb)}.$$

(v) If $u_r = \sin\{a + (r-1)b\}$ we must change the cosines in the numerator of (iv) into sines.

(vi) If $u_r = x^{r-1} + x^{r+n-1} + x^{r+2n-1} + \dots$ ad inf., is

$$(1-x^n)^{-1}.$$

27. The solution of the partial differential equation

$$\begin{vmatrix} D_1, & D_2 & \dots & D_n \\ D_n, & D_1 & \dots & D_{n-1} \\ \dots & \dots & \dots & \dots \\ D_2, & D_3 & \dots & D_1 \end{vmatrix} u = 0,$$

where $$D_r = \frac{d}{dx_r},$$

is $$u = \Sigma F(x_2 - \omega x_1, \quad x_3 - \omega^2 x_1 \dots x_n - \omega^{n-1} x_1),$$

the functions being arbitrary and the summation extending to all values of ω being roots of the equation $x^n - 1 = 0$.

28. If in an orthosymmetrical determinant of order n (VIII. 20),

$$a_k = \frac{(1-q^\alpha)(1-q^{\alpha+1})\dots(1-q^{\alpha+k-2})}{(1-q^\gamma)(1-q^{\gamma+1})\dots(1-q^{\gamma+k-2})},$$

the value of the determinant is equal to

$$\left(\frac{1-q^\alpha}{1-q^\gamma}\right)^{n-1}\left(\frac{1-q^{\alpha+1}}{1-q^{\gamma+1}}\right)^{n-2}\dots\left(\frac{1-q^{\alpha+n-2}}{1-q^{\gamma+n-2}}\right)$$

multiplied by a fraction whose numerator is

$$(-1)^{\frac{n(n-1)}{2}} q^{\frac{n(n-1)(n-2)}{3}} (1-q)^{n-1}(1-q^2)^{n-2}\dots(1-q^{n-1})$$
$$\times (q^\gamma - q^\alpha)^{n-1}(q^{\gamma+1} - q^\alpha)^{n-2}\dots(q^{\gamma+n-2} - q^\alpha),$$

and denominator

$$(1-q^\gamma)(1-q^{\gamma+1})^2\dots(1-q^{\gamma+n-2})^{n-1}$$
$$\times (1-q^{\gamma+n-1})^{n-1}(1-q^{\gamma+n})^{n-2}\dots(1-q^{\gamma+2n-3}).$$

29. The value of the determinant

$$D = \begin{vmatrix} 0 \quad , & a_1 + a_2, & a_1 + a_3 & \dots \\ a_2 + a_1, & 0 \quad , & a_2 + a_3 & \dots \\ a_3 + a_1, & a_3 + a_2, & 0 & \dots \\ \dots & \dots & \dots & \dots \end{vmatrix} (n \text{ rows}),$$

the elements in the leading diagonal being zero, that in the ith row and jth column $a_i + a_j$, is given by

$$(-1)^n D = 2^n a_1 a_2 \dots a_n \left(1 - n - \tfrac{1}{4} \Sigma \frac{(a_i - a_k)^2}{a_i a_k}\right),$$

where i, k are all duads from 1, 2 ... n.

30. The value of the cubic determinant of order n, such that

$$a_{ijk} = a_i + a_j + a_k, \qquad a_{iii} = 0,$$

is given by

$$\frac{(-1)^n D}{3^n a_1 a_2 \dots a_n} = 1 - n - \tfrac{2}{9} \Sigma \frac{(a_i - a_k)^2}{a_i a_k}.$$

And if

$$a_{ijk} = \cos(a_i + a_j + a_k), \quad a_{iii} = 0,$$

$$\frac{(-1)^{n-1} D}{\cos 3a_1 \cos 3a_2 \dots \cos 3a_n} = n - 1 + 2\Sigma \frac{\cos(a_i + a_k)\sin^2(a_i - a_k)}{\cos 3a_i \cos 3a_k},$$

where i, k are all duads from 1, 2 ... n.

31. If $A = |a_{ik}|$, $B = |b_{ik}|$ are two determinants of orders n and m respectively, we can form a new square array of $(nm)^2$ elements as follows. Repeat the array b_{ik}, n times in a row, and take n such rows, so that B is repeated like the squares on a chess-board. Then multiply each of the elements of that block which stands in the ith row and kth column by a_{ik}. The determinant of the resulting array is equal to $A^m B^n$.

Example:

$$A = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}; \qquad B = \begin{vmatrix} \alpha, & \beta \\ \gamma, & \delta \end{vmatrix},$$

$$\begin{vmatrix} a\alpha, & a\beta, & b\alpha, & b\beta \\ a\gamma, & a\delta, & b\gamma, & b\delta \\ c\alpha, & c\beta, & d\alpha, & d\beta \\ c\gamma, & c\delta, & d\gamma, & d\delta \end{vmatrix} = A^2 B^2.$$

32. If $a, b \dots l$; $\alpha, \beta \dots \lambda$ are any two sets of n quantities, and

$$d_{ik} = (a_i - \alpha_k)^r + (b_i - \beta_k)^r + \dots + (l_i - \lambda_k)^r,$$

prove that

$$\begin{vmatrix} d_{11} & \dots & d_{1s} \\ \dots & \dots & \dots \\ d_{s1} & \dots & d_{ss} \end{vmatrix} = 0, \text{ if } s = n(r-1) + 3,$$

$$\begin{vmatrix} d_{11} & \dots & d_{1s}, & 1 \\ \dots & \dots & \dots & \dots \\ d_{s1} & \dots & d_{ss}, & 1 \\ 1 & \dots & 1 & \end{vmatrix} = 0, \text{ if } s = n(r-1)+2.$$

33. $\Big\{$In this and the next five questions

$$m_k = \frac{m(m-1)(m-2)\dots(m-k+1)}{1 \cdot 2 \cdot 3 \dots k}\Big\}.$$

The determinant

$$\begin{vmatrix} m_p, & m_{p+1} & \dots & m_u \\ (m+1)_p, & (m+1)_{p+1} & \dots & (m+1)_u \\ \dots & \dots & \dots & \dots \\ (m+r-1)_p, & (m+r-1)_{p+1} & \dots & (m+r-1)_u \\ (m+r)_p, & (m+r)_{p+1} & \dots & (m+r)_u \\ (m+r+s)_p, & (m+r+s)_{p+1} & \dots & (m+r+s)_u \\ (m+r+s+1)_p, & (m+r+s+1)_{p+1} & \dots & (m+r+s+1)_u \\ \dots & \dots & \dots & \dots \\ (m+r+s+t)_p, & (m+r+s+t)_{p+1} & \dots & (m+r+s+t)_u \end{vmatrix},$$

where $u = p + r + t + 1$ (the suffixes p, $p+1$, ... u of the rows are consecutive, but m, $m+1$, ... $m+r$, $m+r+s$, ... $m+r+s+t$ form two groups of consecutive numbers), is equal to the product of the two fractions

$$\frac{m_p(m+1)_p \dots (m+r)_p (m+r+s)_p \dots (m+r+s+t)_p}{p_p(p+1)_p \dots u_p}$$
$$\frac{(r+s)_{r+1}(r+s+1)_{r+1} \dots (r+s+t)_{r+1}}{(r+1)_{r+1}(r+2)_{r+1} \dots (r+t+1)_{r+1}}.$$

34. The determinant

$$\begin{vmatrix} m_p, & m_{p+1} & \dots & m_{p+s}, & m_{p+s+v} & \dots & m_{p+s+v+u} \\ (m+1)_p, & (m+1)_{p+1} & \dots & (m+1)_{p+s}, & (m+1)_{p+s+v} & \dots & (m+1)_{p+s+v+u} \\ (m+2)_p, & (m+2)_{p+1} & \dots & (m+2)_{p+s}, & (m+2)_{p+s+v} & \dots & (m+2)_{p+s+v+u} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ (m+r)_p, & (m+r)_{p+1} & \dots & (m+r)_{p+s}, & (m+r)_{p+s+v} & \dots & (m+r)_{p+s+v+u} \end{vmatrix},$$

where $r = s + u + 1$ (the suffixes p, $p+1$, ... $p+s$, $p+s+v$, ... $p+s+v+u$ form two groups of consecutive numbers, while m, $m+1$, ... $m+r$ are consecutive), is equal to the product of the two fractions

$$\frac{m_p(m+1)_p \dots (m+r)_p}{p_p(p+1)_p \dots (p+s)_p (p+s+v)_p \dots (p+s+v+u)_p}$$
$$\frac{(m-p)_{v-1}(m-p+1)_{v-1} \dots (m-p+u)_{v-1}}{(v-1)_{v-1} v_{v-1} (v+1)_{v-1} \dots (v+u-1)_{v-1}}.$$

35. Prove that

$$\begin{vmatrix} x^n, & p_0, & p_1 & \dots & p_{r-1} \\ (x+1)^n, & (p+1)_0, & (p+1)_1 & \dots & (p+1)_{r-1} \\ (x+2)^n, & (p+2)_0, & (p+2)_1 & \dots & (p+2)_{r-1} \\ \dots & \dots & \dots & \dots & \dots \\ (x+r)^n, & (p+r)_0, & (p+r)_1 & \dots & (p+r)_{r-1} \end{vmatrix}$$

vanishes if $n<r$, but is equal to $(-1)^n n!$ if $n=r$. If $n>r$ the determinant reduces to a function of x of order $n-r$.

36. Prove that

$$\begin{vmatrix} x^n, & p_1 & \dots & p_r \\ (x+1)^n, & (p+1)_1 & \dots & (p+1)_r \\ (x+2)^n, & (p+2)_1 & \dots & (p+2)_r \\ \dots & \dots & \dots & \dots \\ (x+r)^n, & (p+r)_1 & \dots & (p+r)_r \end{vmatrix} = (x-p)^n$$

for all positive values of n less than r.

37. Prove that

$$\begin{vmatrix} p_0, & p_1 & \dots & p_{r-1} & n^m \\ (p+1)_0, & (p+1)_1 & \dots & (p+1)_{r-1}, & (n+1)^m \\ (p+2)_0, & (p+2)_1 & \dots & (p+2)_{r-1}, & (n+2)^m \\ \dots & \dots & \dots & \dots & \dots \\ (p+r)_0, & (p+r)_1 & \dots & (p+r)_{r-1}, & (n+r)^m \end{vmatrix} = \Delta^r n^m.$$

38. Prove that the value of the determinant

$$\begin{vmatrix} (m-p)\,m_p, & n\,m_{p+1}, & q\,m_{p+2}, & t\,m_{p+3} \\ (m-p+1)(m+1)_p, & (n+1)(m+1)_{p+1}, & (q+1)(m+1)_{p+2}, & (t+1)(m+1)_{p+3}\dots \\ (m-p+2)(m+2)_p, & (n+2)(m+2)_{p+1}, & (q+2)(m+2)_{p+2}, & (t+2)(m+2)_{p+3}\dots \\ \dots & \dots & \dots & \dots \\ (m-p+r)(m+r)_p, & (n+r)(m+r)_{p+1}, & (q+r)(m+r)_{p+2}, & (t+r)(m+r)_{p+3}\dots \end{vmatrix}$$

is $\dfrac{m_p(m+1)_p \dots (m+r)_p}{p_p\,(p+1)_p \dots (p+r)_p}(m-p)(m-p+1)(m-p+2)\dots(m-p+r)$,

and so is independent of the quantities $n, q, t\dots$

39. If $A=|\,a_{ik}\,|$; $B=|\,b_{ik}\,|$ are two determinants of order n, and

$$f(x)=|\,a_{ik}+x\,b_{ik}\,|,$$

prove that $$f(x)f(-x)=AB\,|\,H_{ik}-K_{ik}x^2\,|,$$

where the quantities H_{ik}, K_{ik} satisfy the equations

$$H_{r1}K_{1r}+H_{r2}K_{2r}+\dots+H_{rn}K_{nr}=1,$$
$$H_{r1}K_{1s}+H_{r2}K_{2s}+\dots+H_{rn}K_{ns}=0.$$

40. With the same notation as in the preceding question, prove that if

$$P(\lambda, \mu) = |\lambda a_{ik} + \mu b_{ik}|,$$

then

$$P(\lambda, \mu) = A \begin{vmatrix} \mu H_{11} + \lambda, & \mu H_{12} & \dots & \mu H_{1n} \\ \mu H_{21}, & \mu H_{22} + \lambda & \dots & \mu H_{2n} \\ \dots & \dots & \dots & \dots \\ \mu H_{n1}, & \mu H_{n2} & \dots & \mu H_{nn} + \lambda \end{vmatrix}$$

$$= B \begin{vmatrix} \lambda K_{11} + \mu, & \lambda K_{12} & \dots & \lambda K_{1n} \\ \lambda K_{21}, & \lambda K_{22} + \mu & \dots & \lambda K_{2n} \\ \dots & \dots & \dots & \dots \\ \lambda K_{n1}, & \lambda K_{n2} & \dots & \lambda K_{nn} + \mu \end{vmatrix}.$$

41. If $F(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n$,
prove that

$$P = \begin{vmatrix} x, & 0, & 0 & \dots & 0, & \dfrac{a_n}{a_0} \\ -1, & x, & 0 & \dots & 0, & \dfrac{a_{n-1}}{a_0} \\ 0, & -1, & x & \dots & 0, & \dfrac{a_{n-2}}{a_0} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0 & \dots & x, & \dfrac{a_2}{a_0} \\ 0, & 0, & 0 & \dots & -1, & \dfrac{a_1}{a_0} + x \end{vmatrix} = \frac{F(x)}{a_0},$$

$$Q = \begin{vmatrix} 1 + \dfrac{a_{n-1}}{a_n} x, & -x, & 0 & \dots & 0, & 0 \\ \dfrac{a_{n-2}}{a_n} x, & 1, & -x & \dots & 0, & 0 \\ \dfrac{a_{n-3}}{a_n} x, & 0, & 1 & \dots & 0, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dfrac{a_1}{a_n} x, & 0, & 0 & \dots & 1, & -x \\ \dfrac{a_0}{a_n} x, & 0, & 0 & \dots & 0, & 1 \end{vmatrix} = \frac{F(x)}{a_n}.$$

If P_{rs}, Q_{rs} be the coefficients of homologous elements in P and Q,

$$a_0 P_{rr} x + a_n Q_{rr} = F(x)$$
$$a_0 P_{rs} x + a_n Q_{rs} = 0.$$

Also, if to the elements of P we add the homologous elements of Q multiplied by y, the resulting determinant is equal to

$$\frac{F(x)\,F(y)}{a_0 a_n}.$$

42. Prove the formula for the change of the independent variable in the determinant of n functions

$$\Sigma \pm y_1 \frac{dy_2}{dx}\frac{d^2y_3}{dx^2}\cdots\frac{d^{n-1}y_n}{dx^{n-1}}$$
$$= \left(\frac{dt}{dx}\right)^{\frac{n(n+1)}{2}} \Sigma \pm y_1 \frac{dy_2}{dt}\frac{d^2y_3}{dt^2}\cdots\frac{d^{n-1}y_n}{dt^{n-1}}.$$

43. Let $a_1, a_2, a_3 \ldots$ be a series of n positive numbers, and let s_r be the sum of the divisors of r selected from the terms of this series, this sum being supposed to vanish for all values of r which have no divisors in the above series. Then if

$$D_n = \begin{vmatrix} s_1 s_{n-1} + s_n, & -s_1, & -s_2, & -s_3 & \ldots & -s_{n-2} \\ s_1 s_{n-2} + s_{n-1}, & n-1, & -s_1, & -s_2 & \ldots & -s_{n-3} \\ s_1 s_{n-3} + s_{n-2}, & 0, & n-2, & -s_1 & \ldots & -s_{n-4} \\ s_1 s_{n-4} + s_{n-3}, & 0, & 0, & n-3 & \ldots & -s_{n-5} \\ \hdotsfor{6} \\ s_1 s_1 + s_2, & 0, & 0, & 0 & \ldots & 2 \end{vmatrix},$$

the number of positive integral solutions of the equation

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots = n$$

is

$$\frac{D_n}{n\,!}.$$

44. If s_r is the sum of all the divisors of r, then the determinant

$$\begin{vmatrix} s_{n-1} - s_n, & s_1, & s_2, & s_3 & \ldots & s_{n-3}, & s_{n-2} \\ s_{n-2} - s_{n-1}, & n-1, & s_1, & s_2 & \ldots & s_{n-4}, & s_{n-3} \\ s_{n-3} - s_{n-2}, & 0, & n-2, & s_1 & \ldots & s_{n-5}, & s_{n-4} \\ s_{n-4} - s_{n-3}, & 0, & 0, & n-3 & \ldots & s_{n-6}, & s_{n-5} \\ \hdotsfor{7} \\ s_2 - s_3, & 0, & 0, & 0 & \ldots & 3, & s_1 \\ s_1 - s_2, & 0, & 0, & 0 & \ldots & 0, & 2 \end{vmatrix}$$

is equal to $(-1)^k n\,!$ when n is of the form $\frac{1}{2}(3k^2 \pm k)$, but vanishes for other values of n.

45. Let (m, n) denote the greatest common divisor of the integral numbers m and n; and let $\psi(m)$ be the number of numbers prime to m and not surpassing m; the symmetrical determinant

$$D_m = \Sigma \pm (1, 1)(2, 2) \ldots (m, m)$$

is equal to
$$\psi(1)\,\psi(2)\,\psi(3) \ldots \psi(m).$$

46. If A is a skew determinant of order n in which the principal diagonal elements are equal to z, and A_{ik} its system of first minors, prove that

$$A_{r1}A_{s1} + A_{r2}A_{s2} + \ldots + A_{rn}A_{sn}$$

is equal to Aw_{rs} if n is even, and to $\dfrac{A}{z}w_{rs}$ if n is odd.

47. If $\quad f(x) = x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + a_n = 0$
has for its roots $b_1, b_2 \ldots b_n$, prove that

$$f(x) = \begin{vmatrix} x, & b_1, & b_1 & \ldots b_1, & b_1 \\ b_1, & x, & b_2 & \ldots b_2, & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ b_{n-1}, & b_{n-1}, & b_{n-1} & \ldots x, & b_n \\ 1, & 1, & 1 & \ldots 1, & 1 \end{vmatrix}.$$

And if s_r is the sum of the rth powers of the roots

$$\begin{vmatrix} x^n, & x^{n-1} \ldots x, & 1 \\ s_n, & s_{n-1} \ldots s_1, & s_0 \\ s_{n+1}, & s_n \ldots s_2, & s_1 \\ \ldots & \ldots & \ldots \\ s_{2n-1}, & s_{2n-2} \ldots s_n, & s_{n-1} \end{vmatrix} = (-1)^{\frac{n(n-1)}{2}} \zeta(b_1, b_2 \ldots b_n) f(x).$$

48. Prove that

$$\begin{vmatrix} \alpha_1^r, & \alpha_1^{n-2} \ldots \alpha_1, & 1 \\ \alpha_2^r, & \alpha_2^{n-2} \ldots \alpha_2, & 1 \\ \ldots & \ldots & \ldots \\ \alpha_n^r, & \alpha_n^{n-2} \ldots \alpha_n, & 1 \end{vmatrix} = \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \ldots \alpha_n)\, H_{r-n+1},$$

H_p being the sum of the homogeneous powers and products of order p of

$$\alpha_1, \alpha_2 \ldots \alpha_n.$$

49. If $\qquad a_{rs} = \dfrac{1}{x_s - a_r}, \qquad \alpha_{rs} = \dfrac{1}{(x_s - a_r)^2},$

prove that the value of the determinant of order $2n$

$$\begin{vmatrix} a_{11}, & \alpha_{11} \ldots a_{n1}, & \alpha_{n1} \\ a_{12}, & \alpha_{12} \ldots a_{n2}, & \alpha_{n2} \\ \ldots & \ldots & \ldots \\ a_{1,2n}, & \alpha_{1,2n} \ldots a_{n,2n}, & \alpha_{n,2n} \end{vmatrix}$$

is
$$(-1)^n \frac{\zeta^2(a_1, a_2 \dots a_n)\,\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_{2n})}{[\phi(a_1)\,\phi(a_2)\dots\phi(a_n)]^2},$$
where
$$\phi(x) = (x-x_1)(x-x_2)\dots(x-x_n).$$

50. Prove that the value of the determinant of order $2n+1$ whose ith row is
$$1, \quad \sin a_i, \quad \cos a_i, \quad \sin 2a_i, \quad \cos 2a_i, \dots \sin na_i, \quad \cos na_i,$$
is
$$2^{2n^2}\,\Pi \sin \tfrac{1}{2}(a_i - a_k),$$
where i, k are all duads from $1, 2 \dots n$ $(i > k)$.

Also that the value of the determinant of order $2n$ whose ith row is
$$\sin a_i, \quad \cos a_i, \quad \sin 2a_i, \quad \cos 2a_i \dots \sin na_i, \quad \cos na_i,$$
is
$$2^{2n^2-2n+1}\,\Pi \sin \tfrac{1}{2}(a_i - a_k)\,S,$$
where
$$S = \Sigma \cos \tfrac{1}{2}(a_1 + a_2 + \dots + a_n - a_{n+1} \dots - a_{2n})$$
is formed by dividing the $2n$ angles into two sets of n in all possible ways and taking the cosine of half the difference of the sums of these sets.

51. If
$$A = \begin{vmatrix} \dfrac{1}{a_1 - x_1}, & \dfrac{1}{a_2 - x_1} & \dots & \dfrac{1}{a_n - x_1}, & 1 \\ \dfrac{1}{a_1 - x_2}, & \dfrac{1}{a_2 - x_2} & \dots & \dfrac{1}{a_n - x_2}, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{1}{a_1 - x_{n+1}}, & \dfrac{1}{a_2 - x_{n+1}} & \dots & \dfrac{1}{a_n - x_{n+1}}, & 1 \end{vmatrix},$$
prove that
$$A = (-1)^n \frac{\zeta^{\frac{1}{2}}(a_1, a_2 \dots a_n)\,\zeta^{\frac{1}{2}}(x_1, x_2 \dots x_{n+1})}{\phi(a_1)\,\phi(a_2)\dots\phi(a_n)},$$
where
$$\phi(x) = (x-x_1)(x-x_2)\dots(x-x_{n+1}).$$

If B is the determinant obtained from A by writing $(a_r - x_s)^2$ in place of $(a_r - x_s)$, prove that
$$\frac{A}{B} = \left\{ \begin{matrix} \dfrac{1}{a_1 - x_1}, & \dfrac{1}{a_2 - x_1} & \dots & \dfrac{1}{a_n - x_1}, & 1 \\ \dfrac{1}{a_1 - x_2}, & \dfrac{1}{a_2 - x_2} & \dots & \dfrac{1}{a_n - x_2}, & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dfrac{1}{a_1 - x_{n+1}}, & \dfrac{1}{a_2 - x_{n+1}} & \dots & \dfrac{1}{a_n - x_{n+1}}, & 1 \end{matrix} \right\},$$

the function on the right being formed like a determinant, with all the signs positive instead of alternating.

52. If α, $\beta \ldots \lambda$; α', $\beta' \ldots \lambda'$ are two sets each of n quantities, and C_r is the product of all the binomial coefficients in the expansion of $(1+x)^r$, prove the following equalities:

$$\begin{vmatrix} (\alpha-\alpha')^n, & (\alpha-\beta')^n \ldots (\alpha-\lambda')^n \\ (\beta-\alpha')^n, & (\beta-\beta')^n \ldots (\beta-\lambda')^n \\ \cdots\cdots & \cdots\cdots\cdots\cdots \\ (\lambda-\alpha')^n, & (\lambda-\beta')^n \ldots (\lambda-\lambda')^n \end{vmatrix} = \frac{C_n}{n\,!}\,\zeta^{\frac{1}{2}}(\alpha,\,\beta \ldots \lambda)\,\zeta^{\frac{1}{2}}(\alpha',\,\beta' \ldots \lambda')\,I,$$

where

$$I = \left\{\begin{matrix} \alpha-\alpha', & \alpha-\beta' \ldots \alpha-\lambda' \\ \beta-\alpha', & \beta-\beta' \ldots \beta-\lambda' \\ \cdots\cdots & \cdots\cdots\cdots \\ \lambda-\alpha', & \lambda-\beta' \ldots \lambda-\lambda' \end{matrix}\right\}.$$

If
$$u = (x-\alpha y)(x-\beta y) \ldots (x-\lambda y),$$
$$v = (x-\alpha' y)(x-\beta' y) \ldots (x-\lambda' y),$$
$$I = (12)^n uv,$$

using the notation of invariants,

$$\begin{vmatrix} (\alpha-\alpha')^n \ldots (\alpha-\lambda')^n, & (\alpha-x)^n \\ \cdots\cdots\cdots\cdots & \cdots\cdots \\ (\lambda-\alpha')^n \ldots (\lambda-\lambda')^n, & (\lambda-x)^n \\ (x-\alpha')^n \ldots (x-\lambda')^n & \end{vmatrix} = (-1)^n C_n\,\zeta^{\frac{1}{2}}(\alpha,\,\beta \ldots \lambda)\,\zeta^{\frac{1}{2}}(\alpha',\beta' \ldots \lambda')\,uv,$$

$$\begin{vmatrix} (\alpha-\alpha')^{n+1} \ldots (\alpha-\lambda')^{n+1}, & (\alpha-x)^{n+1} \\ \cdots\cdots\cdots\cdots & \cdots\cdots \\ (\lambda-\alpha')^{n+1} \ldots (\lambda-\lambda')^{n+1}, & (\lambda-x)^{n+1} \\ (x-\alpha')^{n+1} \ldots (x-\lambda')^{n+1} & \end{vmatrix} = (-1)^n \frac{C_{n+1}}{(n+1)\,!}\,\zeta^{\frac{1}{2}}(\alpha,\,\beta \ldots \lambda) \times \zeta^{\frac{1}{2}}(\alpha',\beta' \ldots \lambda')\,I\,.\,uv,$$

where

$$I = \left\{\begin{matrix} \alpha-\alpha' \ldots \alpha-\lambda', & \alpha-x \\ \beta-\alpha' \ldots \beta-\lambda', & \beta-x \\ \cdots\cdots\cdots & \cdots \\ \lambda-\alpha' \ldots \lambda-\lambda', & \lambda-x \\ x-\alpha' \ldots x-\lambda' & \end{matrix}\right\} = -(12)^{n-1}uv.$$

Again,

$$\begin{vmatrix} (\alpha-\alpha')^n \ldots (\alpha-\lambda')^n, & 1 \\ \cdots\cdots\cdots\cdots & \cdots \\ (\lambda-\alpha')^n \ldots (\lambda-\lambda')^n, & 1 \\ 1 \quad \ldots \quad 1 & \end{vmatrix} = (-1)^{n+1} C_n\,\zeta^{\frac{1}{2}}(\alpha \ldots \lambda)\,\zeta^{\frac{1}{2}}(\alpha' \ldots \lambda'),$$

$$\begin{vmatrix} (\alpha-\alpha')^{n+1} & \dots & (\alpha-\lambda')^{n+1}, & 1 \\ \dots & \dots & \dots & \dots \\ (\lambda-\alpha')^{n+1} & \dots & (\lambda-\lambda')^{n+1}, & 1 \\ 1 & \dots & 1 & \end{vmatrix} = (-1)^n \frac{C_{n+1}}{(n+1)!} \zeta^{\frac{1}{2}}(\alpha \dots \lambda)\, \zeta^{\frac{1}{2}}(\alpha' \dots \lambda') I',$$

$$I' = \begin{Bmatrix} \alpha-\alpha' & \dots & \alpha-\lambda', & 1 \\ \dots & \dots & \dots & \dots \\ \lambda-\alpha' & \dots & \lambda-\lambda', & 1 \\ 1 & & 1 & \end{Bmatrix}$$

$$= (12)^{n-1} \frac{du}{dx} \cdot \frac{dv}{dx}.$$

53. Let there be two systems of binary n-tics $u_1 \dots u_n$; $v_1 \dots v_n$ where

$$u_i = a_{0i}\, x^n + n_1 a_{1i} x^{n-1} y + n_2 a_{2i} x^{n-2} y^2 + \dots + a_{ni} y^n,$$

$$v_i = b_{0i}\, x^n + n_1 b_{1i} x^{n-1} y + n_2 b_{2i} x^{n-2} y^2 + \dots + b_{ni} y^n.$$

And let (i, k) be the lineo-linear invariant of u_i and v_k, so that

$$(i, k) = a_{0i} b_{nk} - n_1 a_{1i} b_{n-1k} + n_2 a_{2i} b_{n-2k} - \dots \pm a_{ni} b_{0k}.$$

Prove that

$$\begin{vmatrix} (1, 1) & \dots & (1, n+2) \\ \dots & \dots & \dots \\ (n+2, 1) & \dots & (n+2, n+2) \end{vmatrix} = 0,$$

$$\begin{vmatrix} (1, 1) & \dots & (1, n+1) \\ \dots & \dots & \dots \\ (n+1, 1) & \dots & (n+1, n+1) \end{vmatrix} = C_n \begin{vmatrix} a_{01}, & a_{11} & \dots & a_{n1} \\ \dots & \dots & \dots & \dots \\ a_{0n+1}, & a_{1n+1} & \dots & a_{nn+1} \end{vmatrix} \begin{vmatrix} b_{01}, & b_{11} & \dots & b_{n1} \\ \dots & \dots & \dots & \dots \\ b_{0n+1}, & b_{1n+1} & \dots & b_{nn+1} \end{vmatrix}.$$

54. If $\alpha_1, \alpha_2 \dots \alpha_n$ are the roots of the equation

$$x^n + p_1 x^{n-1} + \dots + p_n = 0,$$

prove that

$$\frac{d(p_1, p_2 \dots p_n)}{d(\alpha_1, \alpha_2 \dots \alpha_n)} = (-1)^{\frac{1}{2}n(n-1)} \zeta^{\frac{1}{2}}(\alpha_1, \alpha_2 \dots \alpha_n).$$

55. If $$u_1 = \frac{x_1}{x_n}, \quad u_2 = \frac{x_2}{x_n} \dots u_{n-1} = \frac{x_{n-1}}{x_n},$$

x_n being a function of $x_1, x_2 \dots x_{n-1}$ given by

$$x_1^2 + x_2^2 + \dots + x_{n-1}^2 + x_n^2 = 1,$$

prove that

$$\frac{d(u_1, u_2 \dots u_{n-1})}{d(x_1, x_2 \dots x_{n-1})} = \frac{1}{x_n^{n+1}}.$$

56. If $u_n = (x+y+z)^n + (x-y-z)^n + (-x+y-z)^n + (-x-y+z)^n$,
prove that the Hessian of u_n is

$$u_{2-n}(x^4+y^4+z^4-2x^2y^2-2y^2z^2-2z^2x^2)^{n-2}$$

multiplied by a numerical factor.

57. If
$$F = u_1 u_2 \ldots u_n,$$
where $u_1, u_2 \ldots u_n$ are linear functions of the n variables $x_1, x_2 \ldots x_n$, prove that

$$F^2 H(\log F) = (-1)^n \left[\frac{d(u_1, u_2 \ldots u_n)}{d(x_1, x_2 \ldots x_n)}\right]^2.$$

Also that

$$\begin{vmatrix} F, & \dfrac{dF}{dx_1} & \cdots & \dfrac{dF}{dx_n} \\ \dfrac{dF}{dx_1}, & \dfrac{d^2F}{dx_1^2} & \cdots & \dfrac{d^2F}{dx_1 dx_n} \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{dF}{dx_n}, & \dfrac{d^2F}{dx_n dx_1} & \cdots & \dfrac{d^2F}{dx_n^2} \end{vmatrix} = (-1)^n F^{n-1} \left[\frac{d(u_1 \ldots u_n)}{d(x_1 \ldots x_n)}\right]^2.$$

58. If u_1, u_2, u_3 be three functions of x, y, and if

$$v_1 = \frac{d(u_2, u_3)}{d(x, y)}, \quad v_2 = \frac{d(u_3, u_1)}{d(x, y)}, \quad v_3 = \frac{d(u_1, u_2)}{d(x, y)},$$

$$w_1 = \frac{d(v_2, v_3)}{d(x, y)}, \text{ \&c.,}$$

prove that

$$\frac{w_1}{u_1} = \frac{w_2}{u_2} = \frac{w_3}{u_3}.$$

59. If u_1, u_2, u_3, u_4 are four functions of x, y, and if

$$v_1 = \begin{vmatrix} \dfrac{d^2u_2}{dx^2}, & \dfrac{d^2u_3}{dx^2}, & \dfrac{d^2u_4}{dx^2} \\ \dfrac{d^2u_2}{dxdy}, & \dfrac{d^2u_3}{dxdy}, & \dfrac{d^2u_4}{dxdy} \\ \dfrac{d^2u_2}{dy^2}, & \dfrac{d^2u_3}{dy^2}, & \dfrac{d^2u_4}{dy^2} \end{vmatrix}$$

and v_2, v_3, v_4 similar determinants formed from u_3, u_4, u_1, &c., then

from v_1, v_2, v_3, v_4 we can form four new functions w_1, w_2, w_3, w_4 in the same way as we obtained $v_1 \dots v_4$ from $u_1 \dots u_4$. Prove that

$$\frac{w_i}{u_i} = \mu \begin{vmatrix} \frac{d^3u_1}{dx^3}, & \frac{d^3u_2}{dx^3}, & \frac{d^3u_3}{dx^3}, & \frac{d^3u_4}{dx^3} \\ \frac{d^3u_1}{dx^2dy}, & \frac{d^3u_2}{dx^2dy}, & \frac{d^3u_3}{dx^2dy}, & \frac{d^3u_4}{dx^2dy} \\ \frac{d^3u_1}{dx\,dy^2}, & \frac{d^3u_2}{dx\,dy^2}, & \frac{d^3u_3}{dx\,dy^2}, & \frac{d^3u_4}{dx\,dy^2} \\ \frac{d^3u_1}{dy^3}, & \frac{d^3u_2}{dy^3}, & \frac{d^3u_3}{dy^3}, & \frac{d^3u_4}{dy^3} \end{vmatrix}$$

where μ is a numerical factor.

60. For the n^2 functions u_{ik} $(i, k = 1, 2 \dots n)$ of the variables $x_1, x_2 \dots x_n$, prove that the cubic determinant whose elements are

$$\frac{du_{ik}}{dx_j} \qquad (i, j, k = 1, 2 \dots n)$$

is a covariant.

61. For the n functions $u_1 \dots u_n$ of the variables $x_1 \dots x_n$, prove that the cubic determinant whose elements are

$$\frac{d^2u_i}{dx_j dx_k} \qquad (i, j, k = 1, 2 \dots n)$$

is a covariant.

62. If the function u of the variables $x_1 \dots x_n$ be transformed by the linear substitution

$$x_i = b_{i1}y_1 + b_{i2}y_2 + \dots + b_{in-1}y_{n-1}$$

to a function v of $n-1$ variables, prove that

$$H(v) = - \begin{vmatrix} 0, & B_1 & \dots & B_n \\ B_1, & u_{11} & \dots & u_{1n} \\ \dots & \dots & \dots & \dots \\ B_n, & u_{n1} & \dots & u_{nn} \end{vmatrix},$$

where $u_{ik} = \dfrac{d^2u}{dx_i dx_k}$, and $(-1)^i B_i$ is the determinant obtained by suppressing the ith row in the array formed by the quantities b_i.

63. If $$u = \Sigma a_{ik}x_ix_k \qquad (i, k = 1, 2 \dots n),$$
and

$$D_r = \begin{vmatrix} a_{11} & \dots & a_{1r} \\ \dots & \dots & \dots \\ a_{r1} & \dots & a_{rr} \end{vmatrix},$$

prove that the substitution

$$x_r = y_r + \frac{1}{D_r}\frac{dD_{r+1}}{da_{rr+1}}y_{r+1} + \dots + \frac{1}{D_{n-1}}\frac{dD_n}{da_{rn}}y_n$$

reduces the given quadric to the sum of the n squares

$$u = \Sigma \frac{D_r}{D_{r-1}} y_r^2 \qquad (r = 1,\ 2 \dots n).$$

64. If u and v are two n-ary quadrics and U, V their reciprocals, prove that we can by the same linear substitution change u into AV and v into BU; A and B are the discriminants of u and v. The determinant C of the substitution is the geometric mean between the discriminants of U and V. If C be regarded as the discriminant of a quadric W, we can by the same linear substitution reduce the three quadrics U, V, W to the sum of squares. The coefficient of any term in W so transformed is the geometric mean between the homologous coefficients in U and V.

65. If to the leading elements of the determinant of an orthogonal substitution of order n we add the quantities a_1, $a_2 \dots a_n$, or the quantities $\frac{1}{a_1}$, $\frac{1}{a_2} \dots \frac{1}{a_n}$, the resulting determinants are equal if

$$a_1 a_2 \dots a_n = 1.$$

66. If c_{ik} are the coefficients of an orthogonal substitution (modulus unity) of order n, prove that

$$D = \begin{vmatrix} c_{11}-1, & c_{12} & \dots & c_{1n} \\ c_{21}\ , & c_{22}-1 & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}\ , & c_{n2} & \dots & c_{nn}-1 \end{vmatrix}$$

is equal to zero if n is odd; but if n is even its value is

$$2^n \frac{[A]}{A},$$

where A is the skew determinant from which the orthogonal substitution is derived, and $[A]$ the same determinant with the elements in the leading diagonal zero.

If D_{ii} is the coefficient of one of the leading terms in D, prove that when n is even

$$2D_{ii} = -D.$$

67. If
$$|c_{ik}| = \epsilon$$
is the determinant of an orthogonal substitution, the equation
$$\begin{vmatrix} c_{11}+x, & c_{12} & \dots & c_{1n} \\ c_{21}, & c_{22}+x & \dots & c_{2n} \\ \dots & \dots & \dots & \dots \\ c_{n1}, & c_{n2} & \dots & c_{nn}+x \end{vmatrix} = 0$$
is a reciprocal one. If n is odd it has one real root $-\epsilon$; if n is even and $\epsilon = -1$ it has the two real roots ± 1. The rest are all imaginary.

68. The maxima and minima values of
$$u = \Sigma a_{ik} x_i x_k,$$
subject to the conditions
$$v = \Sigma b_{ik} x_i x_k$$
$$c_{11}x_1 + c_{12}x_2 + \dots + c_{1n}x_n = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$c_{n-21}x_1 + c_{n-22}x_2 + \dots + c_{n-2n}x_n = 0$$
are given by the equation
$$\begin{vmatrix} b_{11}u - a_{11}v & \dots & b_{1n}u - a_{1n}v, & c_{11}, & c_{21} & \dots & c_{n-21} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ b_{n1}u - a_{n1}v & \dots & b_{nn}u - a_{nn}v, & c_{1n}, & c_{2n} & \dots & c_{n-2n} \\ c_{11} & \dots & c_{1n} & & & & \\ \dots & \dots & \dots & & & & \\ c_{n-21} & \dots & c_{n-2n} & & & & \end{vmatrix} = 0.$$

69. The values of $x_1, x_2 \dots x_m$ which satisfy the equations
$$a_{11}x_1 + a_{21}x_2 + \dots + a_{m1}x_m = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{1r-1}x_1 + a_{2r-1}x_2 + \dots + a_{mr-1}x_m = 0$$
$$a_{1r}x_1 + a_{2r}x_2 + \dots + a_{mr}x_m = 1$$
$$a_{1r+1}x_1 + a_{2r+1}x_2 + \dots + a_{mr+1}x_m = 0$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{1n}x_1 + a_{2n}x_2 + \dots + a_{mn}x_m = 0$$
and make $x_1^2 + x_2^2 + \dots + x_m^2$ a minimum are
$$\frac{1}{2C}\frac{dC}{da_{1r}}, \quad \frac{1}{2C}\frac{dC}{da_{2r}} \dots \frac{1}{2C}\frac{dC}{da_{mr}},$$
where C is the determinant whose elements are given by
$$c_{ik} = a_{1i}a_{1k} + a_{2i}a_{2k} + \dots + a_{mi}a_{mk}.$$

70. The value of the integral

$$\iint \dots x_i x_j \, dx_1 \, dx_2 \dots dx_n,$$

taken for all values of the variables such that

$$\Sigma a_{ij} x_i x_j = < 1,$$

the quadric being a definite positive form (i.e. incapable of becoming negative), is

$$\frac{(\Gamma \tfrac{1}{2})^n}{\Gamma(\tfrac{1}{2}n + 2)} \frac{A_{ij}}{2A^{\frac{3}{2}}},$$

where $A = | a_{ik} |$ is the discriminant of the quadric.

71. The value of the integral

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \epsilon^{-u} \cos(b_1 x_1 + b_2 x_2 + \dots + b_n x_n) \, dx_1 \, dx_2 \dots dx_n,$$

where

$$u = \Sigma a_{ik} x_i x_k,$$

is

$$\sqrt{\left(\frac{\pi^n}{A}\right)} \, \epsilon^{-\frac{v}{4}},$$

where

$$v = -\frac{1}{A} \begin{vmatrix} 0, & b_1, & b_2 & \dots & b_n \\ b_1, & a_{11}, & a_{12} & \dots & a_{1n} \\ \dots & \dots & \dots & \dots & \dots \\ b_n, & a_{n1}, & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

In this question and the next u is supposed to be incapable of becoming negative.

72. The value of the integral

$$\int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} v \epsilon^{-u} \, dx_1 \, dx_2 \dots dx_n,$$

where

$$v = \Sigma b_{ik} x_i x_k, \qquad u = \Sigma a_{ik} x_i x_k,$$

is

$$\sqrt{\left(\frac{\pi^n}{A^3}\right)} \frac{S}{2},$$

where S is the sum of the n determinants obtained by substituting for each column of A in succession the corresponding column of the discriminant of v.

73. Let $a_1, a_2 \ldots a_{2n+1}$ be $2n+1$ real and different numbers in ascending order of magnitude, and let

$$P(x) = (x-a_1)(x-a_3)\ldots(x-a_{2n+1})$$
$$Q(x) = (x-a_2)(x-a_4)\ldots(x-a_{2n})\,A$$
$$R(x) = P(x)\,Q(x),$$

A being a positive number. Then if

$$K_{rs} = \tfrac{1}{2}\int_{a_{2s-1}}^{a_{2s}} \frac{P(x)\,dx}{(x-a_{2r-1})\sqrt{R(x)}}, \quad L_{rs} = \tfrac{1}{2}\frac{Q(a_{2r-1})}{P'(a_{2r-1})}\int_{a_{2s-1}}^{a_{2s}} \frac{P(x)\,dx}{(x-a_{2r-1})^2\sqrt{R(x)}}$$

(these are the complete Abelian integrals of the first and second species), and if also

$$k_{rs} = \frac{\sqrt{-1}}{2}\int_{a_{2s}}^{a_{2s+1}} \frac{P(x)\,dx}{(x-a_{2r-1})\sqrt{R(x)}}, \quad l_{rs} = \frac{\sqrt{-1}}{2}\frac{Q(a_{2r-1})}{P'(a_{2r-1})}\int_{a_{2s}}^{a_{2s+1}} \frac{P(x)\,dx}{(x-a_{2r-1})^2\sqrt{R(x)}}$$

then

$$D = \begin{vmatrix} K_{11}, & L_{11} & \ldots & K_{n1}, & L_{n1} \\ k_{11}, & l_{11} & \ldots & k_{n1}, & l_{n1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ K_{1n}, & L_{1n} & \ldots & K_{nn}, & L_{nn} \\ k_{1n}, & l_{1n} & \ldots & k_{nn}, & l_{nn} \end{vmatrix} = \left(\frac{\pi}{2}\right)^n.$$

Prove also that

$$\frac{dD}{dK_{r,s}} - \frac{dD}{dK_{r,s-1}} = -\left(\frac{\pi}{2}\right)^{n-1} l_{r,s-1}, \quad \frac{dD}{dL_{r,s}} - \frac{dD}{dL_{r,s-1}} = \left(\frac{\pi}{2}\right)^{n-1} k_{r,s-1}$$

$$\frac{dD}{dk_{r,s}} - \frac{dD}{dk_{r,s-1}} = -\left(\frac{\pi}{2}\right)^{n-1} L_{r,s-1}, \quad \frac{dD}{dl_{r,s}} - \frac{dD}{dl_{r,s-1}} = \left(\frac{\pi}{2}\right)^{n-1} K_{r,s-1}.$$

74. Prove that the value of the continued fraction

$$\frac{a}{a+1-}\,\frac{b}{b+1-}\,\frac{c}{c+1-}\text{ ad inf.}$$

is unity.

75. Prove that the product of the two continued fractions

$$a-1+\frac{1^2}{2(a-1)^2+}\,\frac{3^2}{2(a-1)^2+}\cdots$$

$$a+1+\frac{1^2}{2(a+1)^2+}\,\frac{3^2}{2(a+1)^2+}\cdots$$

is a^2.

76. If u_n is the number of terms in a determinant of order n which do not contain any element from the principal diagonal, prove that

$$u_n = nu_{n-1} + (-1)^n,$$

and hence that $\frac{u_n}{n!}$ is the coefficient of x^n in the expansion of $\frac{\epsilon^{-x}}{1-x}$.

77. If u_n is the number of terms in a symmetrical determinant of order n, prove that

$$u_n - nu_{n-1} + \frac{(n-1)(n-2)}{2} u_{n-3} = 0.$$

Also that $\frac{u_n}{n!}$ is the coefficient of x^n in the expansion of

$$\frac{\epsilon^{\frac{1}{2}x+\frac{1}{4}x^2}}{\sqrt{(1-x)}}.$$

78. If $[1 . 3 . 5 \ldots (2n-1)] u_n$ is the number of terms in a skew determinant of order $2n$, prove that

$$u_n = (2n-1) u_{n-1} - (n-1) u_{n-2}.$$

Shew also that $\frac{u_n}{2^n n!}$ is the coefficient of x^n in the expansion of

$$\sqrt[4]{\left\{\frac{\epsilon^x}{1-x}\right\}}.$$

79. If A is the area of a quadrilateral, the co-ordinates of whose angular points are $(x_1, y_1) \ldots (x_4, y_4)$, then

$$2A = \begin{vmatrix} 1, & 0, & x_1, & y_1 \\ 0, & 1, & x_2, & y_2 \\ 1, & 0, & x_3, & y_3 \\ 0, & 1, & x_4, & y_4 \end{vmatrix} = \begin{vmatrix} x_3 - x_1, & y_3 - y_1 \\ x_4 - x_2, & y_4 - y_2 \end{vmatrix}.$$

The area of a quadrilateral inscribed in a circle in terms of its sides is given by

$$16A = -\begin{vmatrix} -a, & b\ , & c\ , & d \\ b\ , & -a, & d\ , & c \\ c\ , & d\ , & -a, & b \\ d\ , & c\ , & b\ , & -a \end{vmatrix}.$$

80. If the planes

$$a_i x + b_i y + c_i z + d_i = 0 \qquad (i = 1, 2, 3, 4, 5)$$

touch the same sphere, then

$$| a_i, b_i, c_i, d_i, u_i | = 0 \qquad (i = 1, 2 \ldots 5),$$

where

$$u_i^2 = a_i^2 + b_i^2 + c_i^2.$$

81. A quadric of revolution passes through five points $P_1 \dots P_5$, and the distances of these points from a focus are $r_1 \dots r_5$.

If V_1 = volume of tetrahedron $P_2P_3P_4P_5$, &c., prove that

$$V_1r_1 + V_2r_2 + \dots + V_5r_5 = 0.$$

82. Let V, V' be the volumes, A, B, C, D; a, b, c, d the areas of the faces of two tetrahedra whose angular points are numbered 1, 2, 3, 4. Also let P_{ik} be the perpendicular from the point i of the first tetrahedron on the face opposite the point k of the second, and p_{ik} a like quantity for the other tetrahedron. Prove that

$$|P_{ik}| \times |p_{ik}| = \frac{(VV')^4}{ABCD\,abcd} \qquad (i, k = 1, 2, 3, 4).$$

83. If A, B, C, D are the directions of four forces in equilibrium, and if AB is the moment of the lines A and B, &c., prove that

$$\begin{vmatrix} 0, & BA, & CA, & DA \\ AB, & 0, & CB, & DB \\ AC, & BC, & 0, & DC \\ AD, & BD, & CD, & 0 \end{vmatrix} = 0.$$

If a, b, c, d are the magnitudes of the forces

$$a = \sqrt{(BC \,.\, CD \,.\, DB)}, \text{ \&c.}$$

84. In Siebeck's determinant, XVII. 22, prove that

$$\frac{dD}{dd_{ik}} = 288vv',$$

where v is the volume of the tetrahedron formed by the face opposite the point i of the first tetrahedron and the centre of the sphere circumscribing the second tetrahedron, and similarly for v'.

85. If in a system of five points d_{ik} is the square of the line joining the ith and kth points, and r is a sixth point of the system, prove that

$$\begin{vmatrix} d_{r1}^2, & d_{r1}d_{r2} + d_{12} & \dots & d_{r1}d_{r5} + d_{15}, & d_{r1} + 1 \\ d_{r1}d_{r2} + d_{12}, & d_{r2}^2 & \dots & d_{r2}d_{r5} + d_{25}, & d_{r2} + 1 \\ \dots & \dots & \dots & \dots & \dots \\ d_{r1}d_{r5} + d_{15}, & d_{r2}d_{r5} + d_{25} & \dots & d_{r5}^2, & d_{r5} + 1 \\ d_{r1} + 1, & d_{r2} + 1 & \dots & d_{r5} + 1, & 1 \end{vmatrix} = 0.$$

86. If in a system of seven straight lines, m_{ik} is the moment of the ith and kth lines, and r is an eighth line, prove that

$$\begin{vmatrix} m_{r1}^2, & m_{r1}m_{r2}+m_{12} & \dots & m_{r1}m_{r7}+m_{17} \\ m_{r1}m_{r2}+m_{12}, & m_{r2}^2 & \dots & m_{r2}m_{r7}+m_{27} \\ \dots & \dots & \dots & \dots \\ m_{r1}m_{r7}+m_{17}, & m_{r2}m_{r7}+m_{27} & \dots & m_{r7}^2 \end{vmatrix} = 0.$$

87. Having given two tetrahedra whose angular points are marked 1, 2, 3, 4, let d_{ik} denote the square of the distance between the ith point of the first and kth point of the second tetrahedron. Prove the following relations:

(i) For two points P, Q the distances of P from the angular points of the first tetrahedron being a_i, of Q from those of the second b_i, and $d = PQ^2$,

$$\begin{vmatrix} d, & 1, & b_1 & \dots & b_4 \\ 1, & 0, & 1 & \dots & 1 \\ a_1, & 1, & d_{11} & \dots & d_{14} \\ \dots & \dots & \dots & \dots & \dots \\ a_4, & 1, & d_{41} & \dots & d_{44} \end{vmatrix} = 0.$$

(ii) For the point P and a plane, q_i being the distances of the vertices of the second tetrahedron from the plane, p the distance of P from the plane,

$$\begin{vmatrix} p, & 0, & q_1 & \dots & q_4 \\ 1, & 0, & 1 & \dots & 1 \\ a_1, & 1, & d_{11} & \dots & d_{14} \\ \dots & \dots & \dots & \dots & \dots \\ a_4, & 1, & d_{41} & \dots & d_{44} \end{vmatrix} = 0.$$

(iii) For two planes, p_i, q_i being the perpendiculars from the angular points of the tetrahedra on them, ϕ the angle between the planes,

$$\begin{vmatrix} -\frac{1}{2}\cos\phi, & 0, & q_1 & \dots & q_4 \\ 0, & 0, & 1 & \dots & 1 \\ p_1, & 1, & d_{11} & \dots & d_{14} \\ \dots & \dots & \dots & \dots & \dots \\ p_4, & 1, & d_{41} & \dots & d_{44} \end{vmatrix} = 0.$$

88. For a system of six and a second system of five spheres, if p_{ik} is the power of the ith and kth spheres,

$$\begin{vmatrix} 1, & p_{11} & \dots & p_{15} \\ \dots & \dots & \dots & \dots \\ 1, & p_{61} & \dots & p_{65} \end{vmatrix} = 0.$$

properties of such determinants have been discussed by H. Poincaré, Helga v. Koch, and T. Cazzaniga.

T. Muir's *Theory of Determinants in the Historical Order of its Development*, Part I. (London, 1890), gives a most careful and complete account of the progress of the theory down to the year 1841. Dr Muir has also compiled a list of writings on determinants, two parts of which have been published in the *Quart. Journ. of Math.* vols. xviii., xxi.; the third part, coming down to 1900, will shortly appear.

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